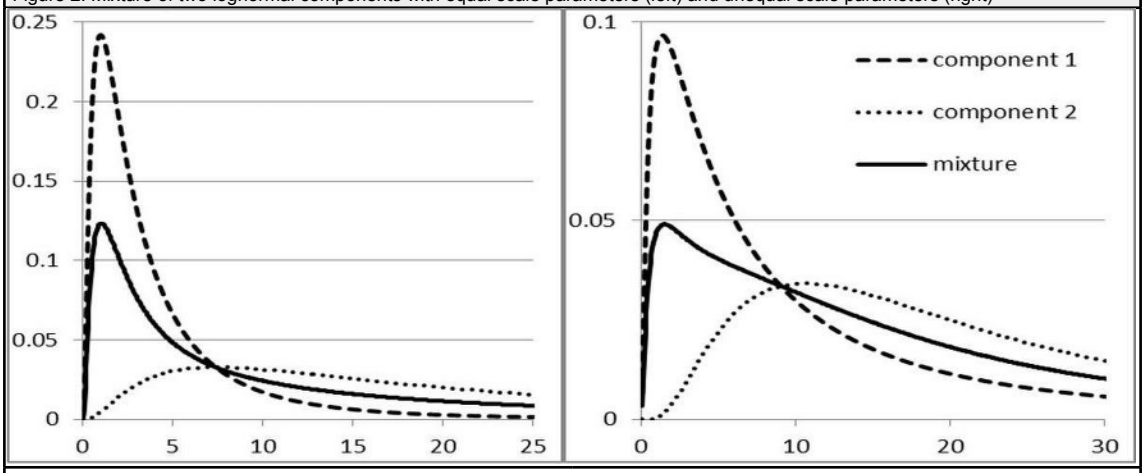
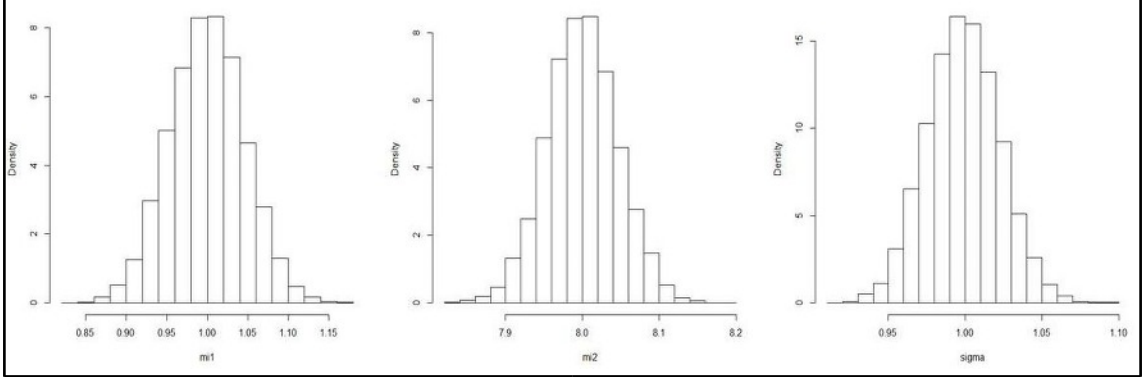
Figure 2: Mixture of two lognormal components with equal scale parameters (left) and unequal scale parameters (right)



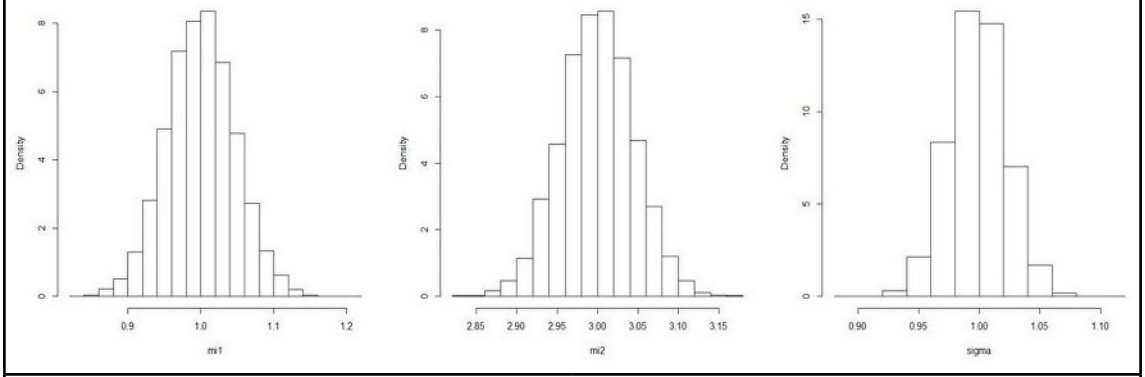
Source: own computations

Figure 3: Distribution of parameters of normal distributions, equal variances: *n*=1,000, 10 % of censored data (Table 1)



Source: own computations

Figure 4: Distribution of parameters lognormal distributions, equal variances: *n*=1,000, 10 % of censored data (Table 3)

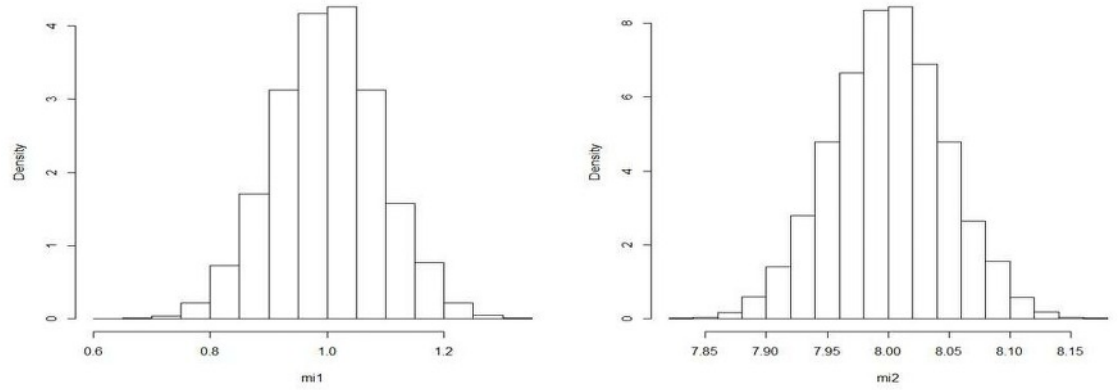


Source: own computations

(Figure 3 for normal and Figure 4 for lognormal component densities). On the horizontal axis estimates of these parameters μ_1, μ_2 and sigma are labelled.

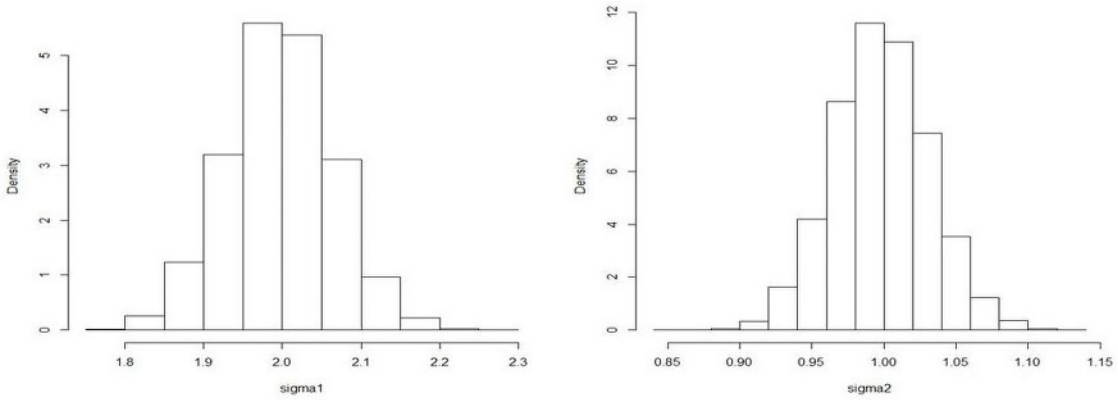
The fits for components with different scale parameters are shown in two figures. In the Figure 5 (normal components) and the Figure 7 (lognormal components) histograms of estimates of parameters μ are shown (labelled mi1 for the first component and μ_2 for the second component). The "true" values are 1 and 8 for the normal components in the Figure 5 and 1 and 3 for the lognormal components in the Figure 7. Estimated scale parameters are given in the Figure 6 (normal component distribution) and Figure 8 (lognormal component distribution) and labelled sigma1 and sigma2 on the horizontal axis. The "true" values in these figures are 2 and 1 for the normal components in the Figure 6 and 1 and 0.64 for the lognormal components in the Figure 8.

Figure 5: Distribution of location parameters of normal distribution, unequal variances, *n*=1,000, 10 % of censored data (Table 2)



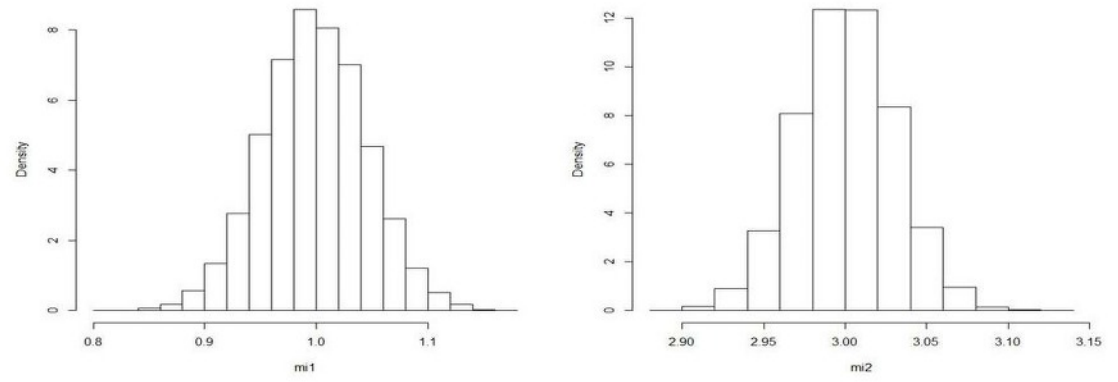
Source: own computations

Figure 6: Distribution of scale parameters of normal distribution, unequal variances, *n*=1,000, 10 % of censored data (Table 2)



Source: own computations

Figure 7: Distribution of location parameters of lognormal distribution, unequal variances, *n*=1,000, 10 % of censored data (Table 4)



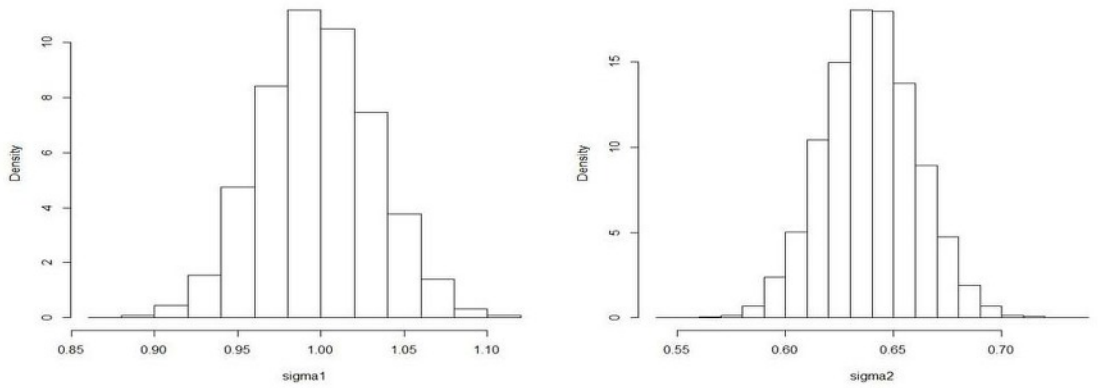
Source: own computations

Conclusion

In this text the problem of estimation of parameters of the finite mixtures of distributions with right censored data was treated. The problem with known (or observable) component membership was discussed; possible methods for models without this information (with artificial components constructed to obtain the best model to fit data) are mentioned together with citations.

It was shown that for relatively large samples (with the use of the simulation) and the percentage of censored data under 50 % the maximum likelihood estimates well approximate unknown values of parameters. The numeric procedure that enables the use of covariances was used to incorporate prior information about parameters (equality of scale parameters) and observed component membership.

Figure 8: Distribution of scale parameters of lognormal distribution, unequal variances, *n*=1,000, 10 % of censored data (Table 2)



Source: own computations

In the histograms based on 10,000 of fits the asymptotic normality of estimates was illustrated.

All simulations and numeric computations were performed in the program R. This program was introduced as the useful tool to be used for estimation problems with various so-called survival distributions (these distributions include normal and lognormal distributions that are used in the text). Moreover, user defined probability distributions can be used in the package Survival through the user defined probability density function called by the procedure.

Acknowledgement

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SPECIAL SURFACES IN R^3

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ABSTRACT

The principal objects of this paper are regular parametrical surfaces in R^3. The method we want to use is based on Weingarten mapping. We suppose that the mapping $x:U\to R^3$, where $(u,v) \in U \subset R^2$ and $x(u,v) \in R^3$, is regular. Symbols x_u and x_v are used in this paper instead of $\partial_u x$, $\partial_v x$ etc. These vectors form the basis of tangent space $T_{x(u,v)}M$ (see Figure 1). On $T_x(M)$ we can construct moving frame $(x_u, x_v, x_u \times x_v)$. Vectors x_u and x_v are tangent vector fields of $T_x(M)$, $x_u \times x_v = n$ is a normal vector field and

$$N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$$

is a unit normal vector field. In this paper we are going to study Gauss and Mean curvature of some classical surfaces with methods based on Weingarten mapping.

JEL CLASSIFICATION & KEYWORDS

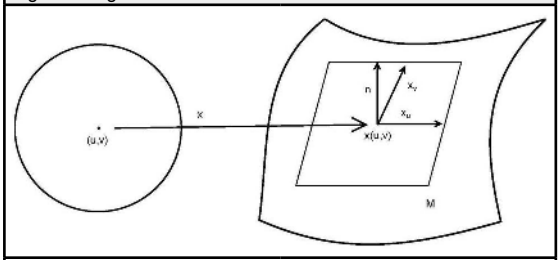
■ C00 ■ WEINGARTEN MAP ■ FIRST AND SECOND FUNDAMENTAL FORMS ■ STRUCTURAL EQUATIONS ■ GAUSSIAN AND MEAN CURVATURE

INTRODUCTION

Let $U \subset R^2$ is an open neighbourhood of a point $(u,v) \in U$ and $x:U\to R^3$ is a regular map (which means that the rank of Jacobian matrix $J(x)(u,v)=2$). A subset $M \subset R^3$ is called a regular two dimensional surface in R^3 if for each point $x = x(u,v)$ there exist an open neighbourhood V of $x(u,v) \in R^3$ and the map $x: U \subset R^2 \to M \cap V$ of an open subset $U \subset R^2$ onto $M \cap V$ is such that

1. x is a differentiable homeomorphism,
2. the differential $dx_q: T_q(U)\to T_{x(q)}(M)$ is injective for all $q \in U$.

Figure 1: Figure



Source: Author

Structural equations

Let U is an open neighbourhood of the point $(u,v) \in R^2$ and $x: U\to R^3$ a regular map. Let $x(U) = M \cap V \cap R^3$, where V is a neighbourhood of the point $x(u,v)$.

$T_x(M)$ is the tangent plane to the surface M at the point $x = x(u,v)$. Tangent vectors x_u and x_v generate vector space $T_x(M)$ (see Figure 1).

Let $N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$ be a unit normal of the surface M. So we have

$$N \cdot x_u = 0,\ N \cdot x_v = 0,\ N \cdot N = 1. \qquad (1)$$

From (1) follows that $N_u \in T_x(M)$ and $N_v \in T_v(M)$.

Remark 1. The first fundamental form of the surface is

$F_1(w_1,w_2) = w_1 \cdot w_2$, where vectors w_1, $w_2 \in T_x(M)$.

In case $w_1 = w_2 = w$, we have

$F_1(w,w) = w \cdot w$, where vectors $w \in T_x(M)$,

$$F_1(w,w) = (ax_u + bx_v)\cdot(ax_u + bx_v) = a^2 g_{11} + 2abg_{12} + b^2 g_{22},$$

where $g_{11} = x_u \cdot x_u$, $g_{12} = x_u \cdot x_v$, $g_{22} = x_v \cdot x_v$.

The first fundamental form can be written in the matrix form

$$F_1(v,v) = (a,b)\cdot\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix},$$

and can be represented by the matrix

$$\mathbf{F_1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

Definition. Let M be a regular surface in R^3 and let N be a unit normal defined in certain neighbourhood of the point $x \in M$. Weingarten mapping is a linear mapping defined by the formula $W(v) = -N_v$, where $N_v \in T_x(M)$ and N_v is the derivative of N in the direction v.

From the previous definition follows $W(u) = -N_u$ and $W(v) = -N_v$.

So we have

$$\begin{aligned} W(x_u)\cdot x_u &= -N_u \cdot x_u, & W(x_u)\cdot x_v &= -N_u \cdot x_v, \\ W(x_v)\cdot x_u &= -N_v \cdot x_u, & W(x_v)\cdot x_v &= -N_v \cdot x_v. \end{aligned} \qquad (2)$$

Formula (1) gives

$$\begin{aligned} (N\cdot x_u)_u &= N_u \cdot x_u + N \cdot x_{uu} = 0, \\ (N\cdot x_u)_v &= N_v \cdot x_u + N \cdot x_{uv} = 0, \\ (N\cdot x_v)_u &= N_u \cdot x_v + N \cdot x_{vu} = 0, \\ (N\cdot x_v)_v &= N_v \cdot x_v + N \cdot x_{vv} = 0. \end{aligned} \qquad (3)$$

From (3) follows

$$N \cdot x_{uv} = -N_v \cdot x_u = W(x_v)\cdot x_u,$$

and

$$N \cdot x_{vu} = -N_u \cdot x_v = W(x_u)\cdot x_v,$$

which means

$$W(x_v)\cdot x_u = W(x_u)\cdot x_v. \qquad (4)$$

As x_u and x_v is the basis of $T_x(M)$ we have

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$$- N_u = a_{11}x_u + a_{12}x_v,$$
$$- N_v = a_{21}x_u + a_{22}x_v. \quad (5)$$

From (2) , (3) and (5) follows

$$\begin{aligned} L_{11} &= W(x_u)\cdot x_u = -N_u \cdot x_u = a_{11}\cdot g_{11} + a_{12}\cdot g_{12},\\ L_{12} &= W(x_u)\cdot x_v = -N_u \cdot x_v = a_{11}\cdot g_{12} + a_{12}\cdot g_{22},\\ L_{21} &= W(x_v)\cdot x_u = -N_v \cdot x_u = a_{21}\cdot g_{11} + a_{22}\cdot g_{12},\\ L_{22} &= W(x_v)\cdot x_v = -N_v \cdot x_v = a_{21}\cdot g_{12} + a_{22}\cdot g_{22}. \end{aligned} \quad (6)$$

The equation (4) gives $L_{12} = L_{21}$.

The equation (6) can be written in the form

$$\begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\cdot\begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix}.$$

If we denote

$$L = \begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}, G = \begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix},$$

we obtain $\mathbf{L} = \mathbf{A}\cdot\mathbf{G}$ or $\mathbf{L}\cdot\mathbf{G}^{-1} = \mathbf{A}$, where (7)

$$\mathrm{G}^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}.$$

Remark 2. The second fundamental form of surface is

$F_2(w_1,w_2) = W(w_1)\cdot w_2$, where vectors $w_1, w_2 \in T_x(M)$,

$F_2(w,w) = W(w)\cdot w = W(ax_u+bx_v)\cdot(ax_u+bx_v)$.

Thanks to linearity of W we have

$$\begin{aligned} F_2(u,u) &= \left(aW(x_u)+bW(x_v)\right)\cdot\left(ax_u+bx_v\right) =\\ &= a^2W(x_u)\cdot x_u + abW(x_v)\cdot x_u + abW(x_u)\cdot x_v + b^2W(x_v)\cdot x_v =\\ &= a^2\left(-N_u\cdot x_u\right) + ab\left(-N_v\cdot x_u\right) + ab\left(-N_u\cdot x_v\right) + b^2\left(-N_v\cdot x_v\right). \end{aligned}$$

So we have

$$F_2(u,u) = a^2W(x_u)\cdot x_u + 2abW(x_u)\cdot x_v + b^2W(x_v)\cdot x_v = a^2L_{11} + 2abL_{12} + b^2L_{22}.$$

The second fundamental form can be expressed in matrix form

$$F_2(u,u) = (a,b)\cdot\begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}\cdot\begin{pmatrix} a\\ b\end{pmatrix}.$$

The second fundamental form can be represented by the matrix

$$\mathbf{F_2} = \begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}.$$

The equation (7) gives

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}\cdot\begin{pmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}.$$

After a short calculation we obtain

$$\begin{aligned} a_{11} &= \frac{L_{11}g_{22} - L_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}, \qquad a_{12} = \frac{-L_{11}g_{12} + L_{12}g_{11}}{g_{11}g_{22} - g_{12}^2},\\ a_{21} &= \frac{L_{12}g_{22} - L_{22}g_{12}}{g_{11}g_{22} - g_{12}^2}, \qquad a_{22} = \frac{-L_{12}g_{12} + L_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}. \end{aligned} \quad (8)$$

From equations (8) follows, that Weingarten mapping can be represented by the matrix

$$\boldsymbol{W} = \begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12}\\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22}\end{pmatrix}.$$

Mean curvature is

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{tr}\quad \mathbf{W} = \frac{1}{2}\frac{\boldsymbol{L}_{11}\boldsymbol{g}_{22} - 2\boldsymbol{L}_{12}\boldsymbol{g}_{12} + \boldsymbol{L}_{22}\boldsymbol{g}_{11}}{\boldsymbol{g}_{11}\boldsymbol{g}_{22} - \boldsymbol{g}_{12}^2}$$

and Gaussian curvature is

$$K = det\ \mathbf{W} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{det\ \mathbf{F}_2}{det\ \mathbf{F}_1}$$

Example 1. Local parameterization of sphere $S^2 \subset R^3$ is $x(u,v) \to (r\cos v\cos u, r\cos v\sin u, r\sin v)$ where $(u,v) \in (0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})$.

The tangent vectors x_u and x_v are

$$x_u = (-r\cos v\sin u, r\cos v\cos u, 0) = r\cos v(-\sin u, \cos u, 0),$$
$$x_v = (-r\sin v\cos u, -r\sin v\sin u, r\cos v) = r(-\sin v\cos u, -\sin v\sin u, \cos v),$$
$$n = r^2(\cos u\cos v, \sin u\cos v, \sin v)$$

and the unit normal $N = (\cos u \cos v, \sin u \cos v, \sin v)$.

We have

$$N_u = (-\sin u\cos v, \cos u\cos v, 0),$$
$$N_v = (-\cos u\sin v, -\sin u\sin v, \cos v).$$

From (6) follows

$$-N_u\cdot x_u = a_{11}\cdot g_{11} + a_{12}\cdot g_{12},$$
$$-N_u\cdot x_v = a_{11}\cdot g_{12} + a_{12}\cdot g_{22}, \quad (9)$$
$$-N_v\cdot x_u = a_{21}\cdot g_{11} + a_{22}\cdot g_{12},$$
$$-N_v\cdot x_v = a_{21}\cdot g_{12} + a_{22}\cdot g_{22}, \quad (10)$$

Substituting into (9) we obtain

$$N_u\cdot x_u = (-\sin u\cos v, \cos u\cos v, 0)\cdot(-r\cos v\sin u, r\cos v\cos u, 0) = r\cos^2 v$$

and

$$r\cos^2 v = -a_{11}r^2\cos^2 v - a_{12}r^2\cos v(\sin v\sin u\cos u - \sin v\sin u\cos u) = -a_{11}r^2\cos^2 v,$$
$$N_u\cdot x_u = r\cos^2 v, \quad g_{12} = x_u\cdot x_v = 0, \quad g_{22} = x_v\cdot x_v = r^2,$$
$$N_u\cdot x_v = 0 = -a_{11}\cdot 0 - a_{12}\cdot r^2,$$

from which follows $a_{12} = 0$. So we have

$$r\cos^2 v = a_{11}r^2\cos^2 v \Rightarrow a_{11} = \frac{-1}{r}.$$

Further we have $N_v\cdot x_u = 0$ and $N_v\cdot x_v = r$.

From (10) and from the equations

$$N_u\cdot x_v = 0, \quad g_{22} = x_v\cdot x_v = r^2, g_{11} = x_u\cdot x_u = r^2\cos^2 v,$$
$$N_v\cdot x_u = 0 = -a_{21}\cdot r^2\cdot\cos^2 v - a_{22}\cdot 0,$$
$$N_v\cdot x_v = r = -a_{21}\cdot 0 - a_{22}\cdot r^2,$$

follows $a_{21} = 0$ and $a_{22} = \frac{-1}{r}$.

Weingarten mapping $W(x_u) = \partial_u N$ and $W(x_v) = \partial_v N$ can be represented by the matrix

$$\mathbf{W} = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix} -\frac{1}{r} & 0\\ 0 & -\frac{1}{r}\end{pmatrix}.$$

For the Gaussian curvature $K = det\ \boldsymbol{W}$ and Mean curvature $H = ½\ tr\ \boldsymbol{W}$, we have $K = 1/r^2$ and $H = -1/r$.

Example 2. Local parameterization of the torus $T^2 \subset R^3$ is given by the map

$$x(u,v) = ((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v),$$

where $a > b > 0$, $(u,v) \in (0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})$,

resp. $(u,v) \in (0,2\pi)\times(\frac{\pi}{2},\frac{3\pi}{2})$.