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GEOMETRICAL DESCRIPTION OF MONKEY'S SADDLE

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ABSTRACT

The principal object of this paper is the regular parametric surface M in R^3 defined by the formula $x(u,v) = (u,v,u^3 - 3uv^2)$. The geometrical description methods we are going to use are based on Cartan's moving frame method and on Weingarten map. The studied map $x(u,v) = (u,v,u^3 - 3uv^2)$ is regular.

JEL CLASSIFICATION & KEYWORDS

■ C00 ■ CARTAN'S MOVING FRAME METHOD ■ WEINGARTEN MAP ■ GAUSSIAN CURVATURE ■ MAIN CURVATURE ■ MOVING FRAME

INTRODUCTION

Let $U \subset R^2$ is an open neighbourhood of a point $(u,v) \in U$ and $x: U \to R^3$ is a regular map (which means that the rank of Jacobian matrix $J(x)(u,v) = 2$). A subset $M \subset R^3$ is called regular two dimensional surface in R^3 if for each $x = x(u,v)$ there exist an open neighbourhood V of $x(u,v) \in R^3$ and the map $x: U \subset R^2 \to M \cap V$ of an open subset $U \subset R^2$ onto $M \cap V$. The map x is given by the formula

$$x(u,v) = (u,v,u^3 - 3uv^2).$$

Now we have to construct the moving frame and orthonormal moving frame which is the base for Cartan method. The next method is based on Weingarten mapping.

Cartan´s method

The moving frame has the form

$$x_u = (1,0,3u^2 - 3v^2),\ x_v = (1,0,-6uv),\ n = (-3u^2 + 3v^2, 6uv, 1).$$

Symbols x_u and x_v are used instead of $\partial_u x$, $\partial_v x$ etc. Vectors x_u and x_v form the basis of the tangent space $T_{x(u,v)}(M)$. On $T_x(M)$ we can construct moving frame $(x_u, x_v, x_u \times x_v)$. As the vectors x_u and x_v are tangent vectors of M the $x_u \times x_v = n$ is a normal vector. Vector $N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$ is the unit normal vector.

Orthonormal moving frame has the form

$$E_1 = \left(\frac{1}{\sqrt{1+9\left(u^2-v^2\right)^2}},\ 0,\ \frac{3\left(u^2-v^2\right)}{\sqrt{1+9\left(u^2-v^2\right)^2}} \right),$$

$$E_3 = \left(\frac{-3u^2+3v^2}{\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{6uv}{\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{1}{\sqrt{1+9\left(u^2+v^2\right)^2}} \right),$$

$$E_2 = \left(\frac{18uv\left(u^2-v^2\right)}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{1+9\left(u^2-v^2\right)^2}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{-6uv}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}} \right)^T,$$

Differential dE_1 equals

$$dE_1 = \left(\frac{-18u\left(u^2-v^2\right)}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}},\ 0,\ \frac{6u}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}} \right) du +$$

$$+ \left(\frac{18v\left(u^2-v^2\right)}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}},\ 0,\ \frac{-6v}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}} \right) dv.$$

Differential form $\omega_{1,2}$ is $\omega_{1,2} = dE_1 * E_2 =$

$$= \frac{-18^2u^2v\left(u^2-v^2\right)^2 - 36u^2v}{\left(1+9\left(u^2-v^2\right)^2\right)^2\sqrt{1+9\left(u^2+v^2\right)^2}} du + \frac{18^2uv^2\left(u^2-v^2\right)^2 + 36uv^2}{\left(1+9\left(u^2-v^2\right)^2\right)^2\sqrt{1+9\left(u^2+v^2\right)^2}} dv =$$

$$= \frac{-36u^2v\left[1+9\left(u^2-v^2\right)^2\right]}{\left(1+9\left(u^2-v^2\right)^2\right)^2\sqrt{1+9\left(u^2+v^2\right)^2}} du + \frac{36uv^2\left[1+9\left(u^2-v^2\right)^2\right]}{\left(1+9\left(u^2-v^2\right)^2\right)^2\sqrt{1+9\left(u^2+v^2\right)^2}} dv =$$

$$= \frac{-36u^2v\,du + 36uv^2\,dv}{\left[1+9\left(u^2-v^2\right)^2\right]\sqrt{1+9\left(u^2+v^2\right)^2}}.$$

We have

$$\partial_u E_3 = \left[\frac{-6u\left[1+18v^2(u^2+v^2)\right]}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}}, \frac{6v\left[1-9(u^4-v^4)\right]}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}}, \frac{-18u\left(u^2+v^2\right)}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} \right],$$

$$\partial_u E_3 * E_1 = \frac{-6u\left[1+18v^2(u^2+v^2)\right] - 54u(u^2-v^2)(u^2+v^2)}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{-6u\left[1+18u^2v^2+18v^4+9u^4-9v^4\right]}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{-6u\left[1+9(u^2+v^2)^2\right]}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{-6u}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}},$$

$$\partial_v E_3 = \left\{ \frac{6v\left[1+18u^2(u^2+v^2)\right]}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}}, \frac{6u\left[1+9(u^4-v^4)\right]}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}}, \frac{-18v\left(u^2+v^2\right)}{\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} \right\}$$

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$$\partial_v E_3 * E_1 = \frac{6v\left[1+9(u^2+v^2)^2\right]}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]^{\frac{3}{2}}} = $$

$$= \frac{6v}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}.$$

The differential form ω_{31} is

$$\omega_{31} = dE_3 * E_1 = \frac{-6u\,du + 6v\,dv}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}.$$

Further we try to construct differential form ω_{32}.

$$\partial_u E_3 * E_2 = \frac{6v\left[1-9(u^4-v^4)\right]}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]},$$

$$\partial_v E_3 * E_2 = \frac{6u\left[1+9(u^4-v^4)\right]}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]}.$$

Differential form ω_{32} is

$$\omega_{32} = \partial_u E_3 * E_2 + \partial_v E_3 * E_2 = \frac{6v\left[1-9(u^4-v^4)\right]du + 6u\left[1+9(u^4-v^4)\right]dv}{\sqrt{1+9(u^2-v^2)^2}\left[1+9(u^2+v^2)^2\right]}.$$

Now we will construct forms θ_1 and θ_2

$\theta_1 = E_1\,dx = E_1\,x_u\,du + E_1\,x_v\,dv,$

$\theta_2 = E_2\,dx = E_2\,x_u\,du + E_2\,x_v\,dv,$

$$E_1 x_u du + E_1 x_v dv = \sqrt{1+9(u^2-v^2)^2}\,du - \frac{18uv(u^2-v^2)}{\sqrt{1+9(u^2-v^2)^2}}\,dv,$$

$$E_2 x_u du = \frac{18uv(u^2-v^2) - 18uv(u^2-v^2)}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}\,du = 0,$$

$$E_2 x_v dv = \frac{1+9(u^2-v^2)^2 + 36du^2v^2}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}\,dv =$$

$$= \frac{1+9(u^2+v^2)^2}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}\,dv.$$

The exterior product of forms θ_1 and θ_2 is

$$\theta_1 \wedge \theta_2 = \frac{1+9\left(u^2+v^2\right)^2}{\sqrt{1+9\left(u^2+v^2\right)^2}}\,du \wedge dv = \sqrt{1+9\left(u^2+v^2\right)^2}\,du \wedge dv,$$

from which follows

$$du \wedge dv = \frac{1}{\sqrt{1+9\left(u^2+v^2\right)^2}}\,\theta_1 \wedge \theta_2.$$

For control we have

$$dE_1 = \left(\frac{-18u\left(u^2-v^2\right)}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}},\quad 0,\quad \frac{6u}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}}\right)du +$$

$$+\left(\frac{18v\left(u^2-v^2\right)}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}},\quad 0,\quad \frac{-6v}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}}\right)dv,$$

$$E_3 = \left(\frac{-3u^2+3v^2}{\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{6uv}{\sqrt{1+9\left(u^2+v^2\right)^2}}, \frac{1}{\sqrt{1+9\left(u^2+v^2\right)^2}}\right)$$

$$\partial_u E_1 \cdot E_3 = \frac{18\cdot 3u\left(u^2-v^2\right)^2 + 6u}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,du =$$

$$= \frac{6u\left[1+9\left(u^2-v^2\right)^2\right]}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,du =$$

$$= \frac{6u}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}\,du\ ,$$

$$\partial_v E_1 \cdot E_3 = \frac{-3\cdot 18v\left(u^2-v^2\right)^2 - 6v}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,dv =$$

$$= \frac{-6v\left[1+9\left(u^2-v^2\right)^2\right]}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,dv =$$

$$= \frac{-6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}\ .$$

The result is

$$\omega_{13} = \frac{6u\,du - 6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}\ .$$

The differential of the form $d\omega_{12} = \omega_{13} \wedge \omega_{32}$ is essential for calculation of the Gauss curvature. Let us study the following equations:

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} =$$

$$\frac{6u\,du - 6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}} \wedge \frac{6v\left[1-9(u^4-v^4)\right]du + 6u\left[1+9\left(u^4-v^4\right)\right]dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\left(1+9\left(u^2+v^2\right)^2\right)} =$$

$$= \frac{36u^2\left[1+9\left(u^4-v^4\right)\right]du \wedge dv + 36v^2\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\,du \wedge dv =$$

$$= \frac{36\left[u^2+u^2\cdot 9\left(u^4-v^4\right)+v^2-v^2\cdot 9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\,du \wedge dv =$$

$$= \frac{3\ 6\left[\left(u^2+v^2\right)+\left(u^2-v^2\right)\cdot 9\left(u^2+v^2\right)\left(u^2-v^2\right)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\,du \wedge dv =$$

$$= \frac{36\left(u^2+v^2\right)\left[1+9\left(u^2-v^2\right)^2\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{1+9\left(u^2+v^2\right)^2}}\,\theta_1 \wedge \theta_2 =$$

$$= \frac{36(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^2} = -K,$$

Where K is the Gaussian curvature. We have

$$\omega_{31} \wedge \omega_{32} = K = \frac{-36(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

Weingarten method

Equation $E_i * E_j = \delta_{ij}$, i, j = 1, 2, 3 gives $dE_i * E_j = 0$, which means $\partial_u E_3 \in T_x(M)$, $\partial_v E_3 \in T_x(M)$.

We have

$$\partial_u E_3 \cdot x_u = \frac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}}\ ,$$

$$\partial_u E_3 \cdot x_v = \frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}}\ ,$$

$$\partial_u E_3 = \beta_{11}\cdot x_u + \beta_{12}\cdot x_v ,$$

$$\partial_u E_3\cdot x_u = \beta_{11} x_u\cdot x_u + \beta_{12} x_v\cdot x_u ,$$

$$\partial_u E_3\cdot x_v = \beta_{11} x_u\cdot x_v + \beta_{12} x_v\cdot x_v ,$$

$$\frac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{11}\left[1+9\left(u^2-v^2\right)^2\right] + \beta_{12}\left[-18uv\left(u^2-v^2\right)\right],$$

$$\frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{11}\left[-18uv\left(u^2-v^2\right)^2\right] + \beta_{12}\left[1+36u^2v^2\right].$$

We have system of two equations for β_{11}, β_{12}. Thanks to Cramer's rule we obtain

$$D=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & -18uv(u^2-v^2)\\ -18uv(u^2-v^2) & (1+36u^2v^2)\end{vmatrix} = 1+9\left(u^2+v^2\right)^2.$$

Further we have

$$\beta_{11}=\begin{vmatrix}\dfrac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} & -18uv(u^2-v^2)\\ \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} & 1+36u^2v^2\end{vmatrix}\cdot\frac{1}{D}=$$

$$=\frac{-6u\left(1+36u^2v^2\right)+6v\cdot 18uv\left(u^2-v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{-6u\left[1+36u^2v^2-18u^2v^2+18v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{-6u\left[1+18v^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

$$\beta_{12}=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & \dfrac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}}\\ -18uv(u^2-v^2) & \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}}\end{vmatrix}\cdot\frac{1}{D}=$$

$$=\frac{6v\left[1+9\left(u^2-v^2\right)^2\right]-6u\cdot 18uv(u^2-v^2)}{\sqrt{1+9\left(u^2+v^2\right)^2}\left[1+9\left(u^2+v^2\right)^2\right]}=$$

$$=\frac{6v\left[1+9u^4-18u^2v^2+9v^4-18u^4+18u^2v^2\right]}{\sqrt{1+9\left(u^2+v^2\right)^2}\left[1+9\left(u^2+v^2\right)^2\right]}=$$

$$=\frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

We have

$$\beta_{11}=\frac{-6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

$$\beta_{12}=\frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

Analogically we obtain

$$\partial_v E_3\cdot x_u = \frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3\cdot x_v = \frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3\cdot x_u = \beta_{21}x_u + \beta_{22}x_v ,$$

$$\frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{21}\left[1+9\left(u^2-v^2\right)^2\right] + \beta_{22}\left[-18uv\left(u^2-v^2\right)\right],$$

$$\partial_v E_3\cdot x_v = \frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3\cdot x_v = \beta_{21}x_u\cdot x_v + \beta_{22}x_v\cdot x_v ,$$

$$\frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{21}\left[-18uv(u^2-v^2)\right] + \beta_{22}\left[1+36u^2v^2\right]$$

$$D=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & -18uv(u^2-v^2)\\ -18uv(u^2-v^2) & 1+36u^2v^2\end{vmatrix} = 1+9\left(u^2+v^2\right)^2$$

$$\beta_{21}=\begin{vmatrix}\dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} & -18uv(u^2-v^2)\\ \dfrac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} & 1+36u^2v^2\end{vmatrix}\cdot\frac{1}{1+9\left(u^2+v^2\right)^2}=$$

$$=\frac{6v\left[1+36u^2v^2+18u^2(u^2-v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{6v\left[1+36u^2v^2-18u^2v^2+18u^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{6v\left[1+18u^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

$$\beta_{22}=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}}\\ -18uv\left(u^2-v^2\right) & \dfrac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}}\end{vmatrix}\cdot\frac{1}{1+9\left(u^2+v^2\right)^2}=$$

$$=\frac{6u\left[1+9u^4-18u^2v^2+9v^4+18v^2(u^2-v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{6u\left[1+9u^4+9v^4-18v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

Weingarten map can be represented by the matrix W

$$W=\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix}=\begin{pmatrix}\dfrac{-6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \dfrac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\\ \dfrac{6v\left[1+18u^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \dfrac{6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\end{pmatrix}.$$

As $W(x_u) = -\partial_u E_3$ and $W(x_v) = -\partial_v E$ we obtain:

$$-W=\begin{pmatrix}\dfrac{6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \dfrac{-6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\\ \dfrac{-6v\left[1+18u^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \dfrac{-6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\end{pmatrix}.$$

$$det(-\mathbf{W}) = \frac{-36}{\left[1+9\left(u^2+v^2\right)^2\right]^3} \cdot D_1 \quad \text{Where}$$

$$\begin{aligned} D_1 &= \left[u^2 + 18u^2v^2\left(u^2+v^2\right)\right]\left[1+9\left(u^4-v^4\right)\right] + \\ &\left[v^2 + 18u^2v^2\left(u^2+v^2\right)\right]\left[1-9\left(u^4-v^4\right)\right] = \\ &= u^2 + 18u^2v^2\left(u^2+v^2\right) + 9u^2\left(u^4-v^4\right) \\ &+ 162u^2v^2\left(u^2+v^2\right)\left(u^4-v^4\right) + v^2 + \\ &18u^2v^2\left(u^2+v^2\right) - 9v^2\left(u^4-v^4\right) - \\ &162u^2v^2\left(u^2+v^2\right)\left(u^4-v^4\right) = u^2+v^2 + \\ &36u^2v^2\left(u^2+v^2\right) + 9\left(u^2+v^2\right)\left(u^2-v^2\right)^2 = \\ &= \left(u^2+v^2\right)\left[1+36u^2v^2+9\left(u^2-v^2\right)^2\right] = \\ &= \left(u^2+v^2\right)\left[1+9u^4+18u^2v^2+9v^4\right] = \\ &= \left(u^2+v^2\right)\left[1+9\left(u^2+v^2\right)^2\right]. \end{aligned}$$

so $$det(-\mathbf{W}) = \frac{-36\left(u^2+v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

As was given before, the Gaussian curvature is

$$K = \frac{-36\left(u^2+v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

Further we obtain

$$H = \frac{1}{2}tr(-W) = \frac{1}{2}\frac{6u\left[1+18v^2\left(u^2+v^2\right)-1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} =$$

$$\frac{27u\left[2u^2v^2+2v^4-u^4+v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} = \frac{27u\left[2u^2v^2+3v^4-u^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}$$

$$= \frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

So the trace of the matrix –W is

$$H = \frac{1}{2}tr(-W) = \frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

Conclusion

By the method of moving frame we reached that the result of Gaussian curvature is

$$K = \frac{-36\left(u^2+v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

By the method of Weingarten mapping we reached that the result of Main curvature is

$$H = \frac{1}{2}tr(-W) = \frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

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