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# THE GAUSSIAN AND MEAN CURVATURE OF ONE SPECIAL TYPE OF SURFACES

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## ABSTRACT

In (Kaňka el al., 2009) we studied the Gaussian curvature and Mean curvature of a special surfaces (1) as nonparametrically defined surfaces. There are different ways in which surfaces of type (1) can be parametrized. The aim of this paper is to give formulas for Gaussian and Mean curvature of one type of special surfaces of the form

$$x_1^\alpha + x_2^\alpha + x_3^\alpha = 1, \text{ where } \alpha \in \mathbb{R},\ \alpha > 0,\ \alpha \neq 1. \quad (1)$$

To reach the formulas of Gaussian and Mean Curvature, we use in this remark parametrical description of (1) in the form

$$g(x_1, x_2) = (x_1, x_2, f(x_1, x_2)),$$

where

$$x_3 = f(x_1, x_2) = (1 - x_1^\alpha - x_2^\alpha)^{\frac{1}{\alpha}},\ 1 - x_1^\alpha - x_2^\alpha > 0\,.$$

## JEL CLASSIFICATION & KEYWORDS

■ C00 ■ TANGENT VECTORS ■ UNIT NORMAL VECTOR ■ FIRST FUNDAMENTAL FORM ■ SECOND FUNDAMENTAL FORM ■ GAUSSIAN CURVATURE ■ MEAN CURVATURE ■ WEINGARTEN MAP ■

## INTRODUCTION

The aim of this paper is to give basic geometrical characteristics (Gaussian and Mean curvature) of special surfaces of type (1). First and second fundamental forms and Weingarten map are used to give general formulas of Gaussian curvature and Mean curvature.

### The General Example of Parametrically Defined Special Surfaces (1)

The tangent and normal vectors at the arbitrary point $\boldsymbol{x} = (x_1, x_2, x_3) \in \boldsymbol{\mathcal{S}}$ are:

$$\begin{aligned} \boldsymbol{g}_{x_1} &= (1,\ 0,\ f_{x_1}(x_1, x_2)), \\ \boldsymbol{g}_{x_2} &= (0,\ 1,\ f_{x_2}(x_1, x_2)), \\ \boldsymbol{n} &= (-f_{x_1}(x_1, x_2),\ -f_{x_2}(x_1, x_2),\ 1)\,. \end{aligned}$$

In case of surface (1) the tangent vectors and the normal vector have the form

$$\begin{aligned} \boldsymbol{g}_{x_1} &= \left(1,\ 0,\ \frac{-x_1^{\alpha-1}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{\alpha-1}{\alpha}}}\right), \\ \boldsymbol{g}_{x_2} &= \left(0,\ 1,\ \frac{-x_2^{\alpha-1}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{\alpha-1}{\alpha}}}\right), \end{aligned} \quad (2)$$

$$\boldsymbol{n} = \left(\frac{x_1^{\alpha-1}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{\alpha-1}{\alpha}}},\ \frac{x_2^{\alpha-1}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{\alpha-1}{\alpha}}},\ 1\right). \quad (3)$$

It is also possible to rewrite vectors (2) and (3) in the form

$$\begin{aligned} \boldsymbol{g}_{x_1} &= \left(1,\ 0,\ -\left(\frac{x_1}{x_3}\right)^{\alpha-1}\right), \\ \boldsymbol{g}_{x_2} &= \left(0,\ 1,\ -\left(\frac{x_2}{x_3}\right)^{\alpha-1}\right), \\ \boldsymbol{n} &= \left(\left(\frac{x_1}{x_3}\right)^{\alpha-1},\ \left(\frac{x_2}{x_3}\right)^{\alpha-1},\ 1\right). \end{aligned}$$

The unit normal $\boldsymbol{N}$ has the form

$$\boldsymbol{N} = \frac{1}{(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2})^{\frac{1}{2}}} \cdot (x_1^{\alpha-1},\ x_2^{\alpha-1},\ x_3^{\alpha-1}).$$

The equation $\boldsymbol{N} \cdot \boldsymbol{N} = 1$ gives

$$\begin{aligned} (\boldsymbol{N} \cdot \boldsymbol{N})_{x_1} = 0, \quad \text{and} \quad (\boldsymbol{N} \cdot \boldsymbol{N})_{x_2} = 0 \quad \Longrightarrow \\ \Longrightarrow \quad 2 \cdot \boldsymbol{N} \cdot \boldsymbol{N}_{x_1} = 0, \quad \text{and} \quad 2 \cdot \boldsymbol{N} \cdot \boldsymbol{N}_{x_2} = 0. \end{aligned} \quad (4)$$

From (4) follows that $\boldsymbol{N}_{x_1}, \boldsymbol{N}_{x_2} \in T_x(S)$. We can express the vecors $\boldsymbol{N}_{x_1}$ and $\boldsymbol{N}_{x_2}$ as a linear combination of the bases of $\boldsymbol{T_x(S)}$, where $\boldsymbol{x} = (x_1, x_2, x_3)$. So we have

$$\begin{aligned} \boldsymbol{N}_{x_1} &= a_{11}\, \boldsymbol{g}_{x_1} + a_{12}\, \boldsymbol{g}_{x_2}, \\ \boldsymbol{N}_{x_2} &= a_{21}\, \boldsymbol{g}_{x_1} + a_{22}\, \boldsymbol{g}_{x_2}. \end{aligned} \quad (5)$$

From (5) follows

$$\begin{aligned} \boldsymbol{N}_{x_1} \cdot \boldsymbol{g}_{x_1} &= a_{11}\, g_{11} + a_{12}\, g_{12}, \\ \boldsymbol{N}_{x_1} \cdot \boldsymbol{g}_{x_2} &= a_{11}\, g_{12} + a_{12}\, g_{22}, \\ &\text{or} \\ \boldsymbol{N}_{x_2} \cdot \boldsymbol{g}_{x_1} &= a_{21}\, g_{11} + a_{22}\, g_{12}, \\ \boldsymbol{N}_{x_2} \cdot \boldsymbol{g}_{x_2} &= a_{21}\, g_{12} + a_{22}\, g_{22}, \end{aligned} \quad (6)$$

where functions

$$g_{11} = 1 + \frac{x_1^{2\alpha-2}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{2\alpha-2}{\alpha}}},$$

$$g_{12} = \frac{(x_1 \cdot x_2)^{\alpha-1}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{2\alpha-2}{\alpha}}},$$

$$g_{22} = 1 + \frac{x_2^{2\alpha-2}}{(1 - x_1^\alpha - x_2^\alpha)^{\frac{2\alpha-2}{\alpha}}},$$

which can be also written in the form

$$g_{11} = 1 + \left(\frac{x_1}{x_3}\right)^{2\alpha-2},$$

$$g_{12} = \left(\frac{x_1}{x_3}\right)^{\alpha-1} \cdot \left(\frac{x_2}{x_3}\right)^{\alpha-1},$$

$$g_{22} = 1 + \left(\frac{x_2}{x_3}\right)^{2\alpha-2},$$

are the coefficients of the first fundamenal form

$$g_{11}\, \mathrm{dx}_1^2 + 2g_{12}\, \mathrm{dx}_1\, \mathrm{dx}_2 + g_{22}\, \mathrm{dx}_2^2.$$

On the other hand we have

$$\boldsymbol{N}\cdot\boldsymbol{g}_{x_1} = 0 \Longrightarrow \boldsymbol{N}_{x_1}\cdot\boldsymbol{g}_{x_1} + \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_1} = 0 \Longrightarrow \Longrightarrow \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_1} = -\boldsymbol{N}_{x_1}\cdot\boldsymbol{g}_{x_1},$$

$$\boldsymbol{N}\cdot\boldsymbol{g}_{x_1} = 0 \Longrightarrow \boldsymbol{N}_{x_2}\cdot\boldsymbol{g}_{x_1} + \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_2} = 0 \Longrightarrow \Longrightarrow \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_2} = -\boldsymbol{N}_{x_2}\cdot\boldsymbol{g}_{x_1}$$

and analogically

$$\boldsymbol{N}\cdot\boldsymbol{g}_{x_2} = 0 \Longrightarrow \boldsymbol{N}_{x_2}\cdot\boldsymbol{g}_{x_2} + \boldsymbol{N}\cdot\boldsymbol{g}_{x_2x_2} = 0 \Longrightarrow \Longrightarrow \boldsymbol{N}\cdot\boldsymbol{g}_{x_2x_2} = -\boldsymbol{N}_{x_2}\cdot\boldsymbol{g}_{x_2},$$

$$\boldsymbol{N}\cdot\boldsymbol{g}_{x_2} = 0 \Longrightarrow \boldsymbol{N}_{x_1}\cdot\boldsymbol{g}_{x_2} + \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_2} = 0 \Longrightarrow \Longrightarrow \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_2} = -\boldsymbol{N}_{x_1}\cdot\boldsymbol{g}_{x_2}.$$

The functions

$$G_{11} = \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_1},$$
$$G_{12} = G_{21} = \boldsymbol{N}\cdot\boldsymbol{g}_{x_1x_2},$$
$$G_{22} = \boldsymbol{N}\cdot\boldsymbol{g}_{x_2x_2}$$

are the coefficients of the second fundamental form

$$G_{11}\, \mathrm{dx}_1^2 \ + \ 2G_{12}\, \mathrm{dx}_1\, \mathrm{dx}_2 \ + \ G_{22}\, \mathrm{dx}_2^2.$$

The functions $G_{11}, G_{12}, G_{22}$ have the form

$$G_{11} = \frac{(\alpha-1)\cdot x_1^{(\alpha-2)}\cdot(x_2^\alpha - 1)}{x_3^\alpha\cdot(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^{\frac{1}{2}}},$$

$$G_{22} = \frac{(\alpha-1)\cdot x_2^{(\alpha-2)}\cdot(x_1^\alpha - 1)}{x_3^\alpha\cdot(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^{\frac{1}{2}}},$$

$$G_{12} = \frac{(1-\alpha)\cdot x_1^{(\alpha-1)}\cdot x_2^{(\alpha-1)}}{x_3^\alpha\cdot(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^{\frac{1}{2}}}.$$

Thanks to equations (6) we obtain

$$\begin{aligned} G_{11} &= -a_{11}\,g_{11} - a_{12}\,g_{12},\\ G_{12} &= -a_{11}\,g_{12} - a_{12}\,g_{22},\\ &\text{or}\\ G_{12} &= -a_{21}\,g_{11} - a_{22}\,g_{12},\\ G_{22} &= -a_{21}\,g_{12} - a_{22}\,g_{22}. \end{aligned} \tag{7}$$

Equatons (7) have the form

$$\begin{pmatrix} G_{11} & G_{12}\\ G_{12} & G_{22}\end{pmatrix} = -\begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix}\cdot\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix},$$

or finaly the form

$$\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix} = -\begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix}^{-1}\cdot\begin{pmatrix} G_{11} & G_{12}\\ G_{12} & G_{22}\end{pmatrix},$$

which means

$$\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix} = = \frac{-1}{g_{11}\cdot g_{22}-g_{12}^2}\begin{pmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}\cdot\begin{pmatrix} G_{11} & G_{12}\\ G_{12} & G_{22}\end{pmatrix}. \tag{8}$$

From (8) we can obtain the real form of functions $a_{11}, a_{12}, a_{21}, a_{22}$ :

$$a_{11} = \frac{G_{11}\cdot g_{22} - G_{12}\cdot g_{12}}{g_{12}^2 - g_{11}\cdot g_{22}},$$
$$a_{21} = \frac{G_{12}\cdot g_{22} - G_{22}\cdot g_{12}}{g_{12}^2 - g_{11}\cdot g_{22}},$$
$$a_{12} = \frac{-G_{11}\cdot g_{12} + G_{12}\cdot g_{11}}{g_{12}^2 - g_{11}\cdot g_{22}},$$
$$a_{22} = \frac{-G_{12}\cdot g_{12} + G_{22}\cdot g_{11}}{g_{12}^2 - g_{11}\cdot g_{22}}.$$

Substituing into (5) we obtain

$$\begin{aligned} -\boldsymbol{N}_{x_1} &= \frac{G_{11}\cdot g_{22} - G_{12}\cdot g_{12}}{g_{11}\cdot g_{22} - g_{12}^2}\,\boldsymbol{g}_{x_1} + \\ &+ \frac{-G_{11}\cdot g_{12} + G_{12}\cdot g_{11}}{g_{11}\cdot g_{22} - g_{12}^2}\,\boldsymbol{g}_{x_2},\\ -\boldsymbol{N}_{x_2} &= \frac{G_{12}\cdot g_{22} - G_{22}\cdot g_{12}}{g_{11}\cdot g_{22} - g_{12}^2}\,\boldsymbol{g}_{x_1} + \\ &+ \frac{-G_{12}\cdot g_{12} + G_{22}\cdot g_{11}}{g_{11}\cdot g_{22} - g_{12}^2}\,\boldsymbol{g}_{x_2}. \end{aligned} \tag{9}$$

The Weingarten map defined for a regular surfaces $\boldsymbol{S}$ by the formula

$$\boldsymbol{W}(\boldsymbol{v_p}) \ = \ -\boldsymbol{N_v},$$

where $\boldsymbol{v_p} \in \boldsymbol{T_p}(\boldsymbol{S})$ and $\boldsymbol{N}$ is a unit normal defined in a neighbourhood of a point $p \in \boldsymbol{S}$ and $\boldsymbol{N_v}$ is the derivative with respect to $\boldsymbol{v_p}$. So we have

$$\boldsymbol{W}(\boldsymbol{g_{x_1}}) \ = \ -\boldsymbol{N_{x_1}}, \quad \text{and} \quad \boldsymbol{W}(\boldsymbol{g_{x_2}}) \ = \ -\boldsymbol{N_{x_2}},$$

The Gaussian curvature $K$ equals to the determinant $det\,\boldsymbol{W}$ which means the determinant

$$K = det\, G$$

where

$$G = \begin{pmatrix} \dfrac{G_{11}\cdot g_{22} - G_{12}\cdot g_{12}}{g_{11}\cdot g_{22} - g_{12}^2} & \dfrac{-G_{11}\cdot g_{12} + G_{12}\cdot g_{11}}{g_{11}\cdot g_{22} - g_{12}^2}\\ \dfrac{G_{12}\cdot g_{22} - G_{22}\cdot g_{12}}{g_{11}\cdot g_{22} - g_{12}^2} & \dfrac{-G_{12}\cdot g_{12} + G_{22}\cdot g_{11}}{g_{11}\cdot g_{22} - g_{12}^2}\end{pmatrix}.$$

So we have

$$K \ = \ \frac{G_{11}\cdot G_{22} \ - \ G_{12}^2}{g_{11}\cdot g_{22} \ - \ g_{12}^2}.$$

The detailed calculation gives (10).

$$K = \frac{\dfrac{(\alpha-1)^2\ x_1^{\alpha-2}x_2^{\alpha-2}(x_1^\alpha-1)(x_2^\alpha-1)-(\alpha-1)^2x_1^{2\alpha-2}x_2^{2\alpha-2}}{x_3^{2\alpha}(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})}}{\dfrac{x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2}}{x_3^{2\alpha-2}}} = \frac{(\alpha-1)^2(x_1x_2x_3)^{\alpha-2}}{(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^2} \tag{10}$$

$$H = \frac{\left(1+\frac{x_2^{2\alpha-2}}{x_3^{2\alpha-2}}\right)\frac{(1-\alpha)x_1^{\alpha-2}(1-x_2^\alpha)}{x_3^{2\alpha-1}} + 2\frac{(\alpha-1)x_1^{2\alpha-2}x_2^{2\alpha-2}}{x_3^{4\alpha-3}} + \left(1+\frac{x_1^{2\alpha-2}}{x_3^{2\alpha-2}}\right)\frac{(1-\alpha)x_2^{\alpha-2}(1-x_1^\alpha)}{x_3^{2\alpha-1}}}{2\left(\frac{x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2}}{x_3^{2\alpha-2}}\right)^{\frac{3}{2}}}. \tag{11}$$

$$H = \frac{(1-\alpha)\left[(x_1x_2)^{\alpha-2}(x_1^\alpha+x_2^\alpha)+(x_2x_3)^{\alpha-2}(x_2^\alpha+x_3^\alpha)+(x_1x_3)^{\alpha-2}(x_1^\alpha+x_3^\alpha)\right]}{2(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^{\frac{3}{2}}}. \tag{12}$$

The mean curvature equals to the trace of the matrix

$$H = \frac{1}{2}\, tr\ G$$

where

$$G = \begin{pmatrix} \dfrac{G_{11}\cdot g_{22}-G_{12}\cdot g_{12}}{g_{11}\cdot g_{22}-g_{12}^2} & \dfrac{-G_{11}\cdot g_{12}+G_{12}\cdot g_{11}}{g_{11}\cdot g_{22}-g_{12}^2} \\ \dfrac{G_{12}\cdot g_{22}-G_{22}\cdot g_{12}}{g_{11}\cdot g_{22}-g_{12}^2} & \dfrac{-G_{12}\cdot g_{12}+G_{22}\cdot g_{11}}{g_{11}\cdot g_{22}-g_{12}^2} \end{pmatrix}.$$

So we have

$$H \ = \ \frac{G_{11}\cdot g_{22}\ -\ 2G_{12}\cdot g_{12}\ +\ G_{22}\cdot g_{11}}{2(g_{11}\cdot g_{22}\ -\ g_{12}^2)}.$$

In case of special surfaces (1) we obtain (11).

After some calculation can be shown, that the formula for Mean curvature can be written in the form (12).

## Conclusion

The Gaussian curvature of special surfaces (1) is given by formula (10) and Mean curvature is given by formula (11).

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