SOME EXAMPLES OF PRODUCTION FUNCTIONS

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ABSTRACT

The aim of this paper is to give some basic geometrical characteristics of generalized Cobb-Douglas surfaces and some examples of these surfaces. In case of growing returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x, y) = (x, y, Az^\alpha y^\beta),$$

where \(x > 0, y > 0, \alpha + \beta > 1\).

In case of decrease returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x, y) = (x, y, Az^\alpha y^\beta),$$

where \(x > 0, y > 0, \alpha + \beta = 1\).

Analogically in case of constant returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x, y) = (x, y, Az^\alpha y^\beta),$$

where \(x > 0, y > 0, \alpha + \beta < 1\).

The basic tools used in this paper are Weingarten map and first and second fundamental forms.

INTRODUCTION

Let \(U \subset \mathbb{R}^2\) and \(x: U \rightarrow \mathbb{R}^3\) is a map. We say that this map is regular if the Jacobian matrix \(J(x)(u, v)\) has rank 2 for all \((u, v) \in U\). Let us suppose that for every point \(p \in M \subset \mathbb{R}^3\) exist an open set \(U \subset \mathbb{R}^2\), an open set \(V \subset \mathbb{R}^3\), \(p \in V\), and a regular differentiable homeomorphism \(x: U \rightarrow V \cap M\). A subset \(M \subset \mathbb{R}^3\) is called a two-dimensional regular surface in \(\mathbb{R}^3\).

The basic tools used in this paper are Weingarten map and first and second fundamental forms.

Surfaces in \(\mathbb{R}^3\)

As was given in [1] we are going to study smooth surfaces, whose atlas consists of regular maps. The basic tool for our study is the shape operator defined as follows.

Definition 1. Let \(S \subset \mathbb{R}^3\) be a regular surface and let \(n\) be a surface normal to \(S\) defined in a neighborhood of a point \(x \in S\). For a tangent vector \(v \in T_x(S)\) we define

$$\varphi(v) = -n_v.$$

Lemma 1. Let \(S \subset \mathbb{R}^2\) and \(\gamma: U \rightarrow \mathbb{R}^3\) be a regular map. Then

$$\varphi(\gamma_x) = -n_x \quad \text{and} \quad \varphi(\gamma_y) = -n_y.$$  \hspace{1cm} (1)

Proof: For fix \(y_0\), \(\gamma(x, y_0)\) is a curve in \(S\). We have

$$\varphi(\gamma_x(x, y_0)) = \varphi(\gamma'(x, y_0)) = -n_x \gamma'(x, y_0) = -n_x.$$  \hspace{1cm} (2)

Analogically

$$\varphi(\gamma_y(x_0, y)) = -n_y.$$  \hspace{1cm} (3)

The shape operator of a regular surface is self adjoint, i.e.

$$\varphi(v) \cdot w = v \cdot \varphi(w)$$

for all tangent vectors \(v, w \in T_x(S)\).

Remark 1. From equations (1) we have

$$0 = (n \cdot \gamma_x)_x = n_x \cdot \gamma_x + n \cdot \gamma_{xx} \Rightarrow n_x \gamma_x = n \cdot \gamma_{xx},$$

$$0 = (n \cdot \gamma_y)_y = n_y \cdot \gamma_y + n \cdot \gamma_{yy} \Rightarrow n_y \gamma_y = n \cdot \gamma_{yy},$$

$$0 = (n \cdot \gamma_x)_y = n_y \cdot \gamma_x + n \cdot \gamma_{xy} \Rightarrow n_y \gamma_x = n \cdot \gamma_{xy} = n \cdot \gamma_x.$$  \hspace{1cm} (4)

Remark 2. Let \(\gamma: U \rightarrow \mathbb{R}^3\) be a regular map. Let us denote

$$l_{11} = -n_x \cdot \gamma_x = n \gamma_{xx},$$

$$l_{12} = -n_y \cdot \gamma_x = n \gamma_{xy} = -n_x \gamma_y,$$

$$l_{22} = -n_y \cdot \gamma_y = n \gamma_{yy}.$$  \hspace{1cm} (5)

The function \(l_{11}, l_{12}, l_{22}\) are coefficients of the second fundamental form \(F_{II}\) of \(\gamma\).

$$F_{II} = l_{11}dx^2 + 2l_{12}dx\,dy + l_{22}dy^2.$$  \hspace{1cm} (6)

If we denote \(g_{11} = ||\gamma_x||^2, g_{12} = \gamma_x \cdot \gamma_y, g_{22} = ||\gamma_y||^2\), the first fundamental form \(F_I\) can be written in the form

$$F_I = g_{11}dx^2 + 2g_{12}dx\,dy + g_{22}dy^2.$$  \hspace{1cm} (7)

Theorem 1. Let \(\gamma: U \rightarrow \mathbb{R}^3\) be a regular map. Then the shape operator \(\varphi\) is given with respect to the basis \(\gamma_x, \gamma_y \in T_x(S)\) in the form

$$\varphi(\gamma_x) = \frac{g_{22}l_{11} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2} \gamma_x + \frac{g_{11}l_{12} - g_{12}l_{11}}{g_{11}g_{22} - g_{12}^2} \gamma_y,$$

$$\varphi(\gamma_y) = \frac{g_{22}l_{12} - g_{12}l_{22}}{g_{11}g_{22} - g_{12}^2} \gamma_x + \frac{g_{11}l_{22} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2} \gamma_y.$$  \hspace{1cm} (8)

Remark 3. As \(\gamma\) is a regular map and \(\gamma_x\) and \(\gamma_y\) are linearly independent we have

$$\varphi(\gamma_x) = \alpha_1 \gamma_x + \alpha_2 \gamma_y = -n_x,$$

$$\varphi(\gamma_y) = \alpha_1 \gamma_x + \alpha_2 \gamma_y = -n_y,$$  \hspace{1cm} (9)
for functions \( \alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22} \) which we need to compute. From (1) and (3) we have
\[
\begin{align*}
    l_{11} &= -n_x \gamma_x = g_{11} \alpha_{11} + g_{12} \alpha_{21}, \\
    l_{12} &= -n_x \gamma_y = g_{12} \alpha_{11} + g_{22} \alpha_{21}, \\
    l_{12} &= -n_y \gamma_x = g_{12} \alpha_{12} + g_{22} \alpha_{22}, \\
    l_{22} &= -n_y \gamma_y = g_{12} \alpha_{12} + g_{22} \alpha_{22}.
\end{align*}
\]
Equations (4) can be written in the form
\[
\begin{pmatrix}
    l_{11} & l_{12} \\
    l_{12} & l_{22}
\end{pmatrix} =
\begin{pmatrix}
    g_{11} & g_{12} \\
    g_{12} & g_{22}
\end{pmatrix}^{-1}
\begin{pmatrix}
    \alpha_{11} & \alpha_{12} \\
    \alpha_{21} & \alpha_{22}
\end{pmatrix},
\]
or
\[
\begin{pmatrix}
    \alpha_{11} & \alpha_{12} \\
    \alpha_{21} & \alpha_{22}
\end{pmatrix} =
\begin{pmatrix}
    g_{11} & g_{12} \\
    g_{12} & g_{22}
\end{pmatrix}^{-1}
\begin{pmatrix}
    l_{11} & l_{12} \\
    l_{12} & l_{22}
\end{pmatrix}.
\]
So we have
\[
\begin{pmatrix}
    \alpha_{11} & \alpha_{12} \\
    \alpha_{21} & \alpha_{22}
\end{pmatrix} =
\frac{1}{g_{11}g_{22} - g_{12}^2}
\begin{pmatrix}
    g_{22}l_{11} - g_{12}l_{12} \\
    g_{11}l_{12} - g_{12}l_{11}
\end{pmatrix},
\]
Gaussian curvature \( K \), and mean curvature \( H \) are defined by formulas
\[
K = \det A(\varphi) \quad \text{and} \quad H = \frac{1}{2} \text{tr} A(\varphi).
\]
We have
\[
K = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2},
\]
and
\[
H = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}.
\]
Examples of Cobb-Douglas Surfaces in \( \mathbb{R}^3 \)

**Example 1.** In case of growing returns to scale we are to study the Gaussian curvature, mean curvature and principal curvatures of a special type Cobb-Douglas surface of the form
\[
\gamma(x, y) = (x, y, xy), \quad \text{i.e.} \ \alpha + \beta = 2 \quad \text{(see Fig. 1)}.
\]
Solution. We have
\[
\gamma_x = (1, 0, y), \quad \gamma_y = (0, 1, x),
\]
\[
g_{11} = 1 + y^2, \quad g_{12} = xy, \quad g_{22} = 1 + x^2.
\]
The unit normal is
\[
n = \frac{(-y, -x, 1)}{\sqrt{x^2 + y^2 + 1}}.
\]
Further we have
\[
\gamma_{xx} = (0, 0, 0), \quad \gamma_{xy} = (0, 0, 1), \quad \gamma_{yy} = (0, 0, 0).
\]
The equations
\[
l_{11} = n \cdot \gamma_{xx}, \quad l_{12} = n \cdot \gamma_{xy}, \quad l_{22} = n \cdot \gamma_{yy}
\]
gives
\[
l_{11} = 0, \quad l_{12} = \frac{1}{\sqrt{x^2 + y^2 + 1}}, \quad l_{22} = 0.
\]
So we have
\[
K = \frac{-1}{(x^2 + y^2 + 1)^2}, \quad H = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}.
\]
From this equation follows that for all \( x, y \in \mathbb{R} \) the Gaussian curvature is negative, which means that every point of this type of Cobb-Douglas surface \( \gamma(x, y) = (x, y, xy) \) is hyperbolic. Principal curvatures \( k_1 \) and \( k_2 \) can be written in the form
\[
k_1 = \frac{1}{(x^2 + y^2 + 1)^{3/2}} \left[-xy + \sqrt{x^2y^2 + (x^2 + y^2 + 1)} \right]
\]
and
\[
k_2 = \frac{1}{(x^2 + y^2 + 1)^{3/2}} \left[-xy - \sqrt{x^2y^2 + (x^2 + y^2 + 1)} \right].
\]

**Example 2.** In this example we are to study Gaussian curvature, mean curvature and principal curvatures of Cobb-Douglas surface
\[
\gamma(x, y) = (x, y, xy^2), \quad x > 0, \quad y > 0, \quad \alpha + \beta = 3 \quad \text{(see Fig. 2)}.
\]
Solution. The basis of $T_x(S)$ has the form
\[ \gamma_x = (1, 0, y^2), \quad \gamma_y = (0, 1, 2xy), \]
\[ g_11 = 1 + y^4, \quad g_{12} = 2xy^3, \quad g_{22} = 1 + 4x^2y^2, \]
\[ \gamma_{xx} = (0, 0, 0), \quad \gamma_{xy} = (0, 0, 2y), \quad \gamma_{yy} = (0, 0, 2x). \]
The unit normal is \[ n = \frac{(-y^2, -2xy, 1)}{\sqrt{y^4 + 4x^2y^2 + 1}}. \] Further we have
\[ l_{11} = 0, \]
\[ l_{12} = \frac{2y}{\sqrt{y^4 + 4x^2y^2 + 1}}, \]
\[ l_{22} = \frac{2x}{\sqrt{y^4 + 4x^2y^2 + 1}}. \]
The Gauss curvature and mean curvature have the forms
\[ K = \frac{-4y^2}{(y^4 + 4x^2y^2 + 1)^2} \leq 0, \]
\[ H = \frac{x - 3xy^4}{(y^4 + 4x^2y^2 + 1)^{3/2}}. \]
If $y \neq 0$ then $K < 0$ and every such a point of given surface is hyperbolic. Principal curvatures can be written in the form (5).

\[ \lambda = \sqrt{4x^2y^4 + 4y^2x^4 + 1}. \]
The Gaussian curvature and mean curvature can be written in the form
\[ K = \frac{-12x^2y^2}{(4x^2y^4 + 4y^2x^4 + 1)^2}, \]
\[ H = \frac{x^2 + y^2 - 8x^2y^4}{(4x^2y^4 + 4y^2x^4 + 1)^{3/2}}. \]
$K \leq 0$ for all $(x, y) \in \mathbb{R}$. Every point $(x, y) \neq (0, 0)$ is hyperbolic.

![Fig. 3](image)

**Example 3.** In this example we are to study Cobb-Douglas surfaces which are practically used in case of growing returns to scale:

1. $\gamma(x, y) = (x, y, x^2y^2)$, i.e. $\alpha + \beta = 4$ (see Fig. 3).

Solution. The basis of $T_x(S)$ has the form
\[ \gamma_x = (1, 0, 2xy^2), \quad \gamma_y = (0, 1, 2y^2), \]
\[ g_11 = 1 + 4x^2y^4, \quad g_{12} = 4x^3y^3, \quad g_{22} = 1 + 4x^4y^2. \]
The unit normal has the form
\[ n = \frac{(-2xy^2, -2y^2x, 1)}{\sqrt{4x^2y^4 + 4y^2x^4 + 1}}. \]
Functions $l_{11}, l_{12}, l_{22}$ are:
\[ l_{11} = \frac{2y^2}{\lambda}, \quad l_{12} = \frac{4xy}{\lambda}, \quad l_{22} = \frac{2x^2}{\lambda}, \]
\[ \lambda = \sqrt{4x^2y^4 + 4y^2x^4 + 1}. \]

2. $\gamma(x, y) = (x, y, x^2y^3)$, i.e. $\alpha + \beta = 5$ (see Fig. 4).

Solution. The basis of tangent space have the form
\[ \gamma_x = (1, 0, 2xy^3), \quad \gamma_y = (0, 1, 3x^2y^2). \]
Functions $g_{11}, g_{12}, g_{22}$ are
\[ g_11 = 1 + 4x^2y^6, \quad g_{12} = 6x^3y^5, \quad g_{22} = 1 + 9x^4y^4. \]
We have
\[ \gamma_{xx} = (0, 0, 2y^3), \]
\[ \gamma_{xy} = (0, 0, 6xy^2), \]
\[ \gamma_{yy} = (0, 0, 6x^2y). \]
The unit normal has the form
\[ n = \frac{(-2xy^3, -3x^2y^2, 1)}{(4x^2y^6 + 9x^4y^4 + 1)^{1/2}}. \]
The function $l_{11}, l_{12}, l_{22}$ are
\[ l_{11} = \frac{2y^3}{\lambda}, \quad l_{12} = \frac{6xy^2}{\lambda}, \quad l_{22} = \frac{6x^2y}{\lambda}, \]
where
\[ \lambda = \left(4x^2y^6 + 9x^4y^4 + 1\right)^{1/2}. \]
The Gaussian curvature and mean curvature are
\[ K = \frac{-24x^2y^4}{(4x^2y^6 + 9x^4y^4 + 1)^2}, \]
\[ H = \frac{y^3 + 3x^2y - 15x^3y^7}{(4x^2y^6 + 9x^4y^4 + 1)^{3/2}}. \]
So we have $K \leq 0$ and if $(x, y) \neq (0, 0)$ then $K < 0$. Every point of given surface for which $(x, y) \neq (0, 0)$ is hyperbolic.
Example 4. In this example we are to study the general case of Cobb-Douglas surface
\[ \gamma(x, y) = (x, y, x^m y^n), \]
where \( m, n \) are constants and
\[ m > 0, \ n > 0. \]

Solution. The basis of tangent space can be written in the form
\[ \gamma_x = (1, 0, mx^{m-1}y^n), \quad \gamma_y = (0, 1, nxy^{n-1}x^m). \]

Function \( g_{11}, g_{12}, g_{22} \) have the form
\[
\begin{align*}
g_{11} &= 1 + m^2x^{2m-2}y^{2n}, \\
g_{12} &= mn \cdot x^{m-1}y^{n-1}, \\
g_{22} &= 1 + n^2y^{2n-2}x^{2m}.
\end{align*}
\]

The unit normal has the form
\[ n = \frac{(-mx^{m-1}y^n, -nxy^{n-1}x^m, 1)}{\lambda^{1/2}} \]
where
\[ \lambda = m^2x^{2m-2}y^{2n} + n^2y^{2n-2}x^{2m} + 1. \]

Further we have
\[
\begin{align*}
\gamma_{xx} &= (0, 0, m(m-1)x^{m-2}y^n), \\
\gamma_{xy} &= (0, 0, m \cdot nx^{m-1}y^{n-1}), \\
\gamma_{yy} &= (0, 0, n(n-1)y^{n-2}x^m).
\end{align*}
\]

The functions \( l_{11}, l_{12}, l_{22} \) can be written in the form
\[
\begin{align*}
l_{11} &= \frac{m(m-1)x^{m-2}y^n}{\lambda}, \\
l_{12} &= \frac{m \cdot n \cdot x^{m-1}y^{n-1}}{\lambda}, \\
l_{22} &= \frac{n(n-1)y^{n-2}x^m}{\lambda}.
\end{align*}
\]

The Gaussian curvature has the form
\[ K = \frac{mn [1 - (m + n)]x^{2m-2}y^{2n-2}}{(m^2x^{2m-2}y^{2n} + n^2y^{2n-2}x^{2m} + 1)^2}. \]

From this formula can be easily seen that following implications are true:

1. \( m + n > 1 \Rightarrow K \leq 0, \quad (x, y) \neq (0, 0) \Rightarrow K < 0, \)
   which means that every point \((x, y) \neq (0, 0)\) of Cobb-Douglas surface is hyperbolic and principal curvatures \(k_1\) and \(k_2\) have opposite signs.

2. \( m + n = 1 \Rightarrow K = 0, \)
   which means that every point of Cobb-Douglas surface is parabolic. Exactly one of principal curvatures is zero.

3. \( m + n < 1 \Rightarrow K > 0, \quad (x, y) \neq (0, 0) \Rightarrow K > 0, \)
   which means that every point \((x, y) \neq (0, 0)\) is elliptic and principal curvatures \(k_1\) and \(k_2\) have the same sign.

The mean curvature can be written in the form (6).

The explicit calculation of principal curvatures \(k_1\) and \(k_2\) is technically a little difficult and so we omit it.

Conclusion

In this paper some examples of Cobb-Douglas surfaces are given and the Gauss and Mean curvatures are studied. In example (4) the general case of Cobb-Douglas surface is given and general formulas of Gaussian and Mean curvatures are analyzed.

References

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