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SOME EXAMPLES OF PRODUCTION FUNCTIONS

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ABSTRACT

The aim of this paper is to give some basic geometrical characteristics of generalized Cobb-Douglas surfaces and some examples of these surfaces. In case of growing returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,Ax^{\alpha}y^{\beta}), \text{ where } x>0,\ y>0,\ \alpha+\beta>1.$$

In case of decrease returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,Ax^{\alpha}y^{\beta}), \text{ where } \ x>0,\ y>0,$$

$$0<\alpha+\beta<1.$$

Analogically in case of constant returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,Ax^{\alpha}y^{\beta}), \text{ where } x>0,\ y>0,\ \alpha+\beta=1.$$

In connection with this surfaces we are interested in Gaussian curvature, mean curvature and principal curvatures.

JEL CLASSIFICATION & KEYWORDS

■ C00 ■ PRINCIPAL CURVATURE ■ GAUSSIAN CURVATURE ■ MEAN CURVATURE ■ FIRST AND SECOND FUNDAMENTAL FORMS ■ GENERALIZED COBB-DOUGLAS SURFACES ■

INTRODUCTION

Let $U\subset\mathbb{R}^2$ and $x:U\to\mathbb{R}^3$ is a map. We say that this map is regular if the Jacobian matrix $J(x)(u,v)$ has rank 2 for all $(u,v)\in U$. Let us suppose that for every point $p\in M\subset\mathbb{R}^3$ exist an open set $U\subset\mathbb{R}^2$, an open set $V\subset\mathbb{R}^3$, $p\in V$, and a regular differentiable homeomorphism $x:U\to V\cap M$. A subset $M\subset\mathbb{R}^3$ is called a two-dimensional regular surface in $\mathbb{R}^3$.

The basic tools used in this paper are Weingarten map and first and second fundamental forms.

Surfaces in $\mathbb{R}^3$

As was given in [1] we are going to study smooth surfaces, whose atlas consists of regular maps. The basic tool for our study is the shape operator defined as follows.

Definition 1. Let $S\subset\mathbb{R}^3$ be a regular surface and let n be a surface normal to S defined in a neighborhood of a point $x\in S$. For a tangent vector $v\in T_x(S)$ we define

$$\varphi(v)=-n_v.$$

Lemma 1. Let $S\subset\mathbb{R}^2$ and $\gamma:\ U\to\mathbb{R}^3$ be a regular map. Then

$$\varphi(\gamma_x)=-n_x \quad \text{and} \quad \varphi(\gamma_y)=-n_y. \tag{1}$$

Proof: For fix $y_0,\ \gamma(x,y_0)$ is a curve in S. We have

$$\begin{aligned}\varphi(\gamma_x(x,y_0))=\varphi(\gamma'(x,y_0))=-n_{\gamma'(x,y_0)}=\\ =-(n\circ\gamma)'(x)=-n_x.\end{aligned}$$

Analogically

$$\varphi(\gamma_y(x_0,y))=-n_y.$$

Lemma 2. At each point x of a regular surface $S\subset\mathbb{R}^3$, the shape operator is a linear map

$$\varphi:T_x(S)\to T_x(S).$$

The shape operator of a regular surface is self adjoint, i.e.

$$\varphi(v)\cdot w=v\cdot\varphi(w)$$

for all tangent vectors $v,w\in T_x(S)$.

Remark 1. From equations (1) we have

$$\begin{aligned}0&=(n\cdot\gamma_x)_x=n_x\cdot\gamma_x+n\cdot\gamma_{xx}\Rightarrow -n_x\gamma_x=n\cdot\gamma_{xx},\\ 0&=(n\cdot\gamma_y)_y=n_y\cdot\gamma_y+n\cdot\gamma_{yy}\Rightarrow -n_y\gamma_y=n\cdot\gamma_{yy},\\ 0&=(n\cdot\gamma_x)_y=n_y\cdot\gamma_x+n\cdot\gamma_{xy}\Rightarrow -n_y\gamma_x=n\cdot\gamma_{xy}=\\ &=n\cdot\gamma_{yx}.\end{aligned}$$

Remark 2. Let $\gamma:\ U\to\mathbb{R}^3$ be a regular map. Let us denote

$$\begin{aligned}l_{11}&=-\ n_x\cdot\gamma_x=n\gamma_{xx},\\ l_{12}&=-\ n_y\cdot\gamma_x=n\gamma_{xy}=n\gamma_{yx}=-n_x\gamma_y,\\ l_{22}&=-\ n_y\cdot\gamma_y=n\gamma_{yy}.\end{aligned}$$

The function l_{11},l_{12},l_{22} are coefficients of the second fundamental form $\mathbb{F}_{II}$ of γ.

$$\mathbb{F}_{II}=l_{11}dx^2+2l_{12}dx\,dy+l_{22}^2\,dy^2.$$

If we denote $g_{11}=||\gamma_x||^2,\ g_{12}=\gamma_x\cdot\gamma_y,\ g_{22}=||\gamma_y||^2$, the first fundamental form $\mathbb{F}_I$ can be written in the form

$$\mathbb{F}_I=g_{11}dx^2+2g_{12}dx\,dy+g_{22}dy^2.$$

Theorem 1. Let $\gamma:\ U\to\mathbb{R}^3$ be a regular map. Then the shape operator φ is given with respect to the basis $\gamma_x,\ \gamma_y\in T_x(S)$ in the form

$$\begin{aligned}\varphi(\gamma_x)&=\frac{g_{22}l_{11}-g_{12}l_{12}}{g_{11}g_{22}-g_{12}^2}\gamma_x+\frac{g_{11}l_{12}-g_{12}l_{11}}{g_{11}g_{22}-g_{12}^2}\gamma_y,\\ \varphi(\gamma_y)&=\frac{g_{22}l_{12}-g_{12}l_{22}}{g_{11}g_{22}-g_{12}^2}\gamma_x+\frac{g_{11}l_{22}-g_{12}l_{12}}{g_{11}g_{22}-g_{12}^2}\gamma_y.\end{aligned} \tag{2}$$

Remark 3. As γ is a regular map and γ_x and γ_y are lineraly independent we have

$$\begin{aligned}\varphi(\gamma_x)&=\alpha_{11}\gamma_x+\alpha_{21}\gamma_y=-n_x,\\ \varphi(\gamma_y)&=\alpha_{12}\gamma_x+\alpha_{22}\gamma_y=-n_y,\end{aligned} \tag{3}$$

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for functions $\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}$ which we need to compute. From (1) and (3) we have

$$\begin{aligned} l_{11} &= -\, n_x \gamma_x = g_{11}\alpha_{11} + g_{12}\alpha_{21}, \\ l_{12} &= -\, n_x \gamma_y = g_{12}\alpha_{11} + g_{22}\alpha_{21}, \\ l_{12} &= -\, n_y \gamma_x = g_{11}\alpha_{12} + g_{12}\alpha_{22}, \\ l_{22} &= -\, n_y \gamma_y = g_{12}\alpha_{12} + g_{22}\alpha_{22}. \end{aligned} \tag{4}$$

Equations (4) can be written in the form

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}, \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}, \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}.$$

So we have

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}, \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

from which immediately follows (2).

Remark 4. The shape operator can be represented by a matrix

$$\mathbb{A}(\varphi) = \begin{pmatrix} \dfrac{g_{22}l_{11} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2}, & \dfrac{g_{11}l_{12} - g_{12}l_{11}}{g_{11}g_{22} - g_{12}^2} \\ \dfrac{g_{22}l_{12} - g_{12}l_{22}}{g_{11}g_{22} - g_{12}^2}, & \dfrac{g_{11}l_{22} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2} \end{pmatrix}.$$

Gaussian curvature $\mathbb{K}$, and mean curvature $\mathbb{H}$ are defined by formulas $\mathbb{K} = \det \mathbb{A}(\varphi)$ and $\mathbb{H} = \frac{1}{2}\mathrm{tr}\mathbb{A}(\varphi)$. We have

$$\mathbb{K} = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

and

$$\mathbb{H} = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}.$$

Examples of Cobb-Douglas Surfaces in $\mathbb{R}^3$

Example 1. In case of growing returns to scale we are to study the Gaussian curvature, mean curvature and principal curvatures of a special type Cobb-Douglas surface of the form

$$\gamma(x, y) = (x, y, xy), \quad \text{i.e. } \alpha + \beta = 2 \text{ (see Fig. 1).}$$

Solution. We have

$$\begin{aligned} &\gamma_x = (1, 0, y), \quad \gamma_y = (0, 1, x), \\ &g_{11} = 1 + y^2, \quad g_{12} = xy, \quad g_{22} = 1 + x^2. \end{aligned}$$

The unit normal is

$$n = \frac{(-y, -x, 1)}{\sqrt{x^2 + y^2 + 1}}.$$

Further we have

$$\gamma_{xx} = (0, 0, 0), \quad \gamma_{xy} = (0, 0, 1), \quad \gamma_{yy} = (0, 0, 0).$$

The equations

$$l_{11} = n \cdot \gamma_{xx}, \quad l_{12} = n \cdot \gamma_{xy}, \quad l_{22} = n \cdot \gamma_{yy}$$

gives

$$l_{11} = 0, \quad l_{12} = \frac{1}{\sqrt{x^2 + y^2 + 1}}, \quad l_{22} = 0.$$

So we have

$$K = \frac{-1}{(x^2 + y^2 + 1)^2}, \qquad H = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}.$$

From this equation follows that for all $x, y \in \mathbb{R}$ the Gaussian curvature is negative, which means that every point of this type of Cobb-Douglas surface $\gamma(x, y) = (x, y, xy)$ is hyperbolic. Principal curvatures k_1 and k_2 can be written in the form

$$k_1 = \frac{1}{(x^2 + y^2 + 1)^{3/2}} \left[-xy + \sqrt{x^2y^2 + (x^2 + y^2 + 1)} \right]$$

and

$$k_2 = \frac{1}{(x^2 + y^2 + 1)^{3/2}} \left[-xy - \sqrt{x^2y^2 + (x^2 + y^2 + 1)} \right].$$

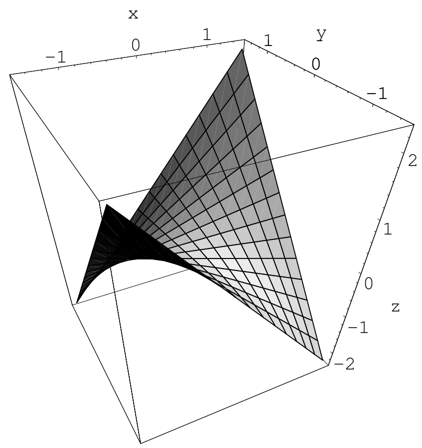

Fig. 1

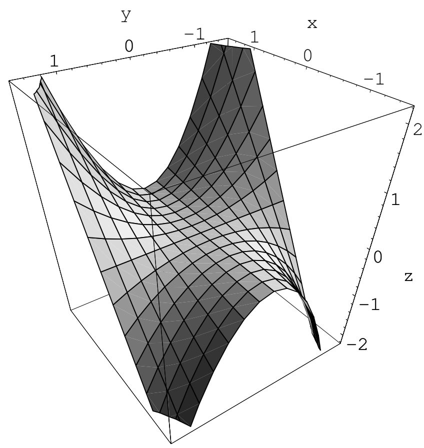

Fig. 2

Example 2. In this example we are to study Gaussian curvature, mean curvature and principal curvatures of Cobb-Douglas surface

$\gamma(x, y) = (x, y, xy^2)$, $x > 0$, $y > 0$, $\alpha + \beta = 3$ (see Fig. 2).

Solution. The basis of $T_x(S)$ has the form

$$\gamma_x = (1, 0, y^2), \quad \gamma_y = (0, 1, 2xy),$$

$$g_{11} = 1 + y^4, \quad g_{12} = 2xy^3, \quad g_{22} = 1 + 4x^2y^2,$$

$$\gamma_{xx} = (0, 0, 0), \quad \gamma_{xy} = (0, 0, 2y), \quad \gamma_{yy} = (0, 0, 2x).$$

The unit normal is $n = \dfrac{(-y^2, -2xy, 1)}{\sqrt{y^4 + 4x^2y^2 + 1}}$. Further we have

$$l_{11} = 0,$$

$$l_{12} = \frac{2y}{\sqrt{y^4 + 4x^2y^2 + 1}},$$

$$l_{22} = \frac{2x}{\sqrt{y^4 + 4x^2y^2 + 1}}.$$

The Gauss curvature and mean curvature have the forms

$$K = \frac{-4y^2}{(y^4 + 4x^2y^2 + 1)^2} \leq 0,$$

$$H = \frac{x - 3xy^4}{(y^4 + 4x^2y^2 + 1)^{3/2}}.$$

If $y \neq 0$ then $K < 0$ and every such a point of given surface is hyperbolic. Principal curvatures can be written in the form (5).

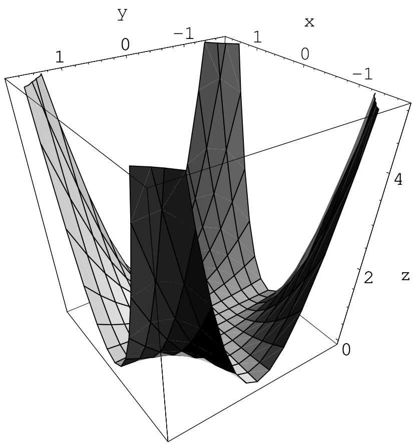

Fig. 3

Example 3. In this example we are to study Cobb-Douglas surfaces which are practically used in case of growing returns to scale:

1. $\gamma(x, y) = (x, y, x^2y^2)$, i.e. $\alpha + \beta = 4$ (see Fig. 3).
   Solution. The basis of $T_x(S)$ has the form

$$\gamma_x = (1, 0, 2xy^2), \quad \gamma_y = (0, 1, 2yx^2),$$

$$g_{11} = 1 + 4x^2y^4, \quad g_{12} = 4x^3y^3, \quad g_{22} = 1 + 4x^4y^2.$$

   The unit normal has the form

$$n = \frac{(-2xy^2, -2yx^2, 1)}{\sqrt{4x^2y^4 + 4y^2x^4 + 1}}.$$

   Functions l_{11}, l_{12}, l_{22} are:

$$l_{11} = \frac{2y^2}{\lambda} \quad l_{12} = \frac{4xy}{\lambda}, \quad l_{22} = \frac{2x^2}{\lambda},$$

   where

$$\lambda = \sqrt{4x^2y^4 + 4y^2x^4 + 1}.$$

   The Gaussian curvature and mean curvature can be written in the form

$$K = \frac{-12x^2y^2}{(4x^2y^4 + 4y^2x^4 + 1)^2},$$

$$H = \frac{x^2 + y^2 - 8x^4y^4}{(4x^2y^4 + 4y^2x^4 + 1)^{3/2}}.$$

   $K \leq 0$ for all $(x, y) \in \mathbb{R}$. Every point $(x, y) \neq (0, 0)$ is hyperbolic.

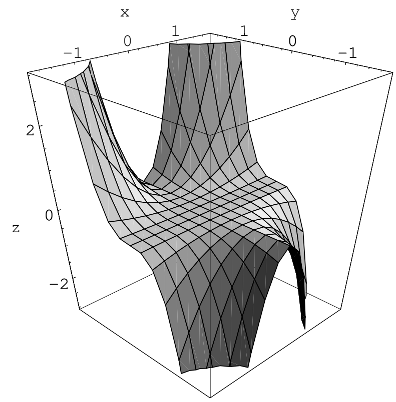

Fig. 4

2. $\gamma(x, y) = (x, y, x^2y^3)$, i.e. $\alpha + \beta = 5$ (see Fig. 4).
   Solution. The basis of tangent space have the form

$$\gamma_x = (1, 0, 2xy^3), \gamma_y = (0, 1, 3x^2y^2).$$

   Functions g_{11}, g_{12}, g_{22} are

$$g_{11} = 1 + 4x^2y^6, \quad g_{12} = 6x^3y^5, \quad g_{22} = 1 + 9x^4y^4.$$

   We have

$$\gamma_{xx} = (0, 0, 2y^3),$$

$$\gamma_{xy} = (0, 0, 6xy^2),$$

$$\gamma_{yy} = (0, 0, 6x^2y).$$

   The unit normal has the form

$$n = \frac{(-2xy^3, -3x^2y^2, 1)}{(4x^2y^6 + 9x^4y^4 + 1)^{1/2}}.$$

   The function l_{11}, l_{12}, l_{22} are

$$l_{11} = \frac{2y^3}{\lambda}, \quad l_{12} = \frac{6xy^2}{\lambda}, \quad l_{22} = \frac{6x^2y}{\lambda},$$

   where

$$\lambda = \left(4x^2y^6 + 9x^4y^4 + 1\right)^{1/2}.$$

   The Gaussian curvature and mean curvature are

$$K = \frac{-24x^2y^4}{(4x^2y^6 + 9x^4y^4 + 1)^2},$$

$$H = \frac{y^3 + 3x^2y - 15x^4y^7}{(4x^2y^6 + 9x^4y^4 + 1)^{3/2}}.$$

   So we have $K \leq 0$ and if $(x, y) \neq (0, 0)$ then $K < 0$. Every point of given surface for which $(x, y) \neq (0, 0)$ is hyperbolic.

$$k_1 = \frac{1}{(y^4+4x^2y^2+1)^{3/2}}\left[x-3xy^4+\sqrt{(x-3xy^4)^2+4y^2(y^4+4x^2y^2+1)}\right],$$
$$k_2 = \frac{1}{(y^4+4x^2y^2+1)^{3/2}}\left[x-3xy^4-\sqrt{(x-3xy^4)^2+4y^2(y^4+4x^2y^2+1)}\right]. \quad (5)$$

$$H = \frac{m(m-1)x^{m-2}y^n + n(n-1)y^{n-2}x^m - mn(m+n)x^{3m-2}y^{3n-2}}{(m^2x^{2m-2}y^{2n}+n^2x^{2m}y^{2n-2}+1)^{3/2}}. \quad (6)$$

Example 4. In this example we are to study the general case of Cobb-Douglas surface

$$\gamma(x,y) = (x, y, x^m y^n),$$

where m, n are constants and

$$m > 0, \; n > 0.$$

Solution. The basis of tangent space can be written in the form

$$\gamma_x = (1, 0, mx^{m-1}y^n), \quad \gamma_y = (0, 1, ny^{n-1}x^m).$$

Function $g_{11},\ g_{12},\ g_{22}$ have the form

$$g_{11} = 1 + m^2x^{2m-2}y^{2n},$$
$$g_{12} = mn \cdot x^{2m-1}y^{2n-1},$$
$$g_{22} = 1 + n^2y^{2n-2}x^{2m}.$$

The unit normal has the form

$$n = \frac{(-mx^{m-1}y^n,\ -ny^{n-1}x^m,\ 1)}{\lambda^{1/2}}$$

where

$$\lambda = m^2x^{2m-2}y^{2n} + n^2y^{2n-2}\cdot x^{2m} + 1.$$

Further we have

$$\gamma_{xx} = (0, 0, m(m-1)x^{m-2}y^n),$$
$$\gamma_{xy} = (0, 0, m\cdot nx^{m-1}y^{n-1}),$$
$$\gamma_{yy} = (0, 0, n(n-1)y^{n-2}x^m).$$

The functions $l_{11},\ l_{12},\ l_{22}$ can be written in the form

$$l_{11} = \frac{m(m-1)x^{m-2}y^n}{\sqrt{\lambda}},$$
$$l_{12} = \frac{m\cdot n\cdot x^{m-1}y^{n-1}}{\sqrt{\lambda}},$$
$$l_{22} = \frac{n(n-1)y^{n-2}x^m}{\sqrt{\lambda}}.$$

The Gaussian curvature has the form

$$K = \frac{mn\,[1-(m+n)]x^{2m-2}y^{2n-2}}{(m^2x^{2m-2}y^{2n}+n^2x^{2m}y^{2n-2}+1)^2}.$$

From this formula can be easily seen that following implications are true:

1. $m+n > 1 \Rightarrow K \le 0, \quad (x,y) \ne (0,0) \Rightarrow K < 0,$
   which means that every point $(x,y) \ne (0,0)$ of Cobb-Douglas surface is hyperbolic and principal curvatures k_1 and k_2 have oposite signs.

2. $m+n = 1 \Rightarrow K = 0,$
   which means that every point of Cobb-Douglas surface is parabolic. Exactly one of principal curvatures is zero.

3. $m+n < 1 \Rightarrow K \ge 0, \quad (x,y) \ne (0,0) \Rightarrow K > 0,$
   which means that every point $(x,y) \ne (0,0)$ is eliptic and principal curvatures k_1 and k_2 have the same sign.

The mean curvature can be written in the form (6).

The explicit calculation of principal curvatures k_1 and k_2 is technically a little difficult and so we omit it.

Conclusion

In this paper some examples of Cobb-Douglas surfaces are given and the Gauss and Mean curvatures are studied. In example (4) the general case of Cobb-Douglas surface is given and general formulas of Gaussian and Mean curvatures are analyzed.

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