ISSN 2045-3345

GEOMETRY OF FUNCTIONS IN ECONOMICS (APPLICATION OF CARTAN'S MOVING FRAME METHOD)

Miloš KAŇKA[1], Eva KAŇKOVÁ[2]

[1] College of Polytechnics, Jihlava, Czech Republic
[2] University of Ecomomics in Prague, Czech Republic

ABSTRACT

Our principal object of study is the geometry of special submanifolds of R^3. The method we are going to use was invented by Darboux and brought to perfection by Cartan. On a Riemannian manifold $(M, \langle,\rangle)$ we define an orthonormal moving frame $(X_1, \ldots, X_n)$ such that $(X_1(p), \ldots, X_n(p)$ is an orthonormal frame for tangent space M_p. The aim of this article is to give geometrical analysis of a special type of Cobb-Douglas surface, especially the formula of Gauss curvature

$$\gamma(x, y) = (x, y, Ax^\alpha y^\beta),$$

where

$$A = 1,\ x > 0, y > 0,\ \alpha = 1 \text{ or } \alpha = 2 \text{ and } \beta = 1.$$

For this purpose we use the Cartan's moving frame method.

JEL CLASSIFICATION & KEYWORDS

■ C00 ■ EXTERIOR PRODUCT ■ EXTERIOR DIFFERENTIATION ■ MAURER-CARTAN STRUCTURAL EQUATIONS ■ GAUSSIAN CURVATURE ■ ORTHONORMAL MOVING FRAME ■ PARAMETRIZED UTILITY SURFACE ■ WEINGARTEN MAP ■

INTRODUCTION

Let $U \subset \mathbb{R}^2$ and $x : U \to \mathbb{R}^3$ is a map. We say that this map is regular if the Jacobian matrix $J(x)(u, v)$ has rank 2 for all $(u, v) \in U$. Let us suppose that for every point $p \in M \subset \mathbb{R}^3$ exist an open set $U \subset \mathbb{R}^2$, an open set $V \subset \mathbb{R}^3$, $p \in V$, and a regular differentiable homeomorphism $x : U \to V \cap M$. A subset $M \subset \mathbb{R}^3$ is called a two-dimensional regular surface in $\mathbb{R}^3$. Let $x(U) \subset V \cap M \subset \mathbb{R}^3$ be a neighbourhood of $p \in M$ such that the restriction $x|U$ is an differentiable homeomorphism into $x(U) \subset V \cap M$ and that it is possible to choose in $x(U)$ an orthonormal moving frame $\{E_1, E_2, E_3\}$ in such a way that E_1, E_2 are tangent to $x(U)$ and E_3 is a non-vanishing normal to $x(U)$. We first discuss the Cartan structural equations for a two-dimensional surface in $\mathbb{R}^3$.

Structural Equations

We first discuss the Cartan structural equations for a two-dimensional surface in $\mathbb{R}^3$. Differentiating a patch $x(u, v)$ we obtain

$$dx = x_u du + x_v dv, \tag{1}$$

where x_u, x_v are tangent vector fields. Let us denote

$$n(u, v) = x_u \times x_v$$

the normal vector field. With respect to the orthonormal moving frame $\{E_1, E_2, E_3\}$ we define forms

$$\theta_i = E_i dx = E_i x_u du + E_i x_v dv,\ i = 1, 2, 3.$$

Since x_u and x_v are tangent to $x(U)$ we have $E_3 \cdot dx = 0$ which implies $\theta_3 = 0$. So we have

$$\theta_1 = E_1 x_u du + E_1 x_v dv,$$
$$\theta_2 = E_2 x_u du + E_2 x_v dv.$$

Each vector $E_i : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable map and the differential

$$dE_i : \mathbb{R}^3 \to \mathbb{R}^3$$

is a linear map. So we may write (using Einstein's notation)

$$dE_i = \omega_{ij} E_j$$

where ω_{ij} are linear forms on $\mathbb{R}^3$ and since E_i are differentiable, ω_{ij} are 9 differentiable forms. So we have

$$\begin{aligned} dE_1 &= \omega_{11}E_1 + \omega_{12}E_2 + \omega_{13}E_3, \\ dE_2 &= \omega_{21}E_1 + \omega_{22}E_2 + \omega_{23}E_3, \\ dE_3 &= \omega_{31}E_1 + \omega_{32}E_2 + \omega_{33}E_3. \end{aligned} \tag{2}$$

Differentiating the equation $E_i \cdot E_j = \delta_{ij}$, where δ_{ij} is the Kronecker's symbol, we obtain

$$dE_i E_j + E_i dE_j = \omega_{ij} + \omega_{ji} = 0.$$

Forms ω_{ij} are antisymmetric

$$\omega_{ii} = 0, \quad \omega_{ij} = -\omega_{ji}. \tag{3}$$

From (2) and (3) we have

$$\begin{aligned} dE_1 &= \qquad\quad \omega_{12}E_2 + \omega_{13}E_3, \\ dE_2 &= -\omega_{12}E_1 \qquad\quad + \omega_{23}E_3, \\ dE_3 &= -\omega_{13}E_1 - \omega_{23}E_2. \end{aligned} \tag{4}$$

Forms dx and dE_i have vanishing exterior derivatives, which implies

$$0 = d^2x = dE_1 \wedge \theta_1 + E_1 d\theta_1 + dE_2 \wedge \theta_2 + E_2 d\theta_2. \tag{5}$$

Substituting (4) into (5) we obtain

$$\begin{aligned} &(\omega_{12}E_2 + \omega_{13}E_3) \wedge \theta_1 + E_1 d\theta_1 + \\ &+ (\omega_{21}E_1 + \omega_{23}E_3) \wedge \theta_2 + E_2 d\theta_2 = 0. \end{aligned} \tag{6}$$

From (6) there immediately follows

$$\begin{aligned} &(d\theta_1 + \omega_{21} \wedge \theta_2)E_1 + \\ &+ (d\theta_2 + \omega_{12} \wedge \theta_1)E_2 + \\ &+ (\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2)E_3 = 0 \end{aligned} \tag{7}$$

The linear independence of vectors E_1, E_2, E_3 and equation (7) gives the following equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2, \tag{8}$$
$$d\theta_2 = \omega_{21} \wedge \theta_1, \tag{9}$$
$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2. \tag{10}$$

RESEARCH JOURNAL OF ECONOMICS, BUSINESS AND ICT

Differentiating (4) gives:

$$0 = d^2E_1 = d\omega_{12}E_2 - \omega_{12} \wedge dE_2 + d\omega_{13}E_3 - \omega_{13} \wedge dE_3,$$

$$d\omega_{12}E_2 - \omega_{12} \wedge (\omega_{21}E_1 + \omega_{23}E_3) + \\ + d\omega_{13}E_3 - \omega_{13} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0,$$

$$(d\omega_{12} - \omega_{13} \wedge \omega_{32})E_2 + \\ +(d\omega_{13} - \omega_{12} \wedge \omega_{23})E_3 = 0. \quad (11)$$

From (11) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}, \\ d\omega_{13} = \omega_{12} \wedge \omega_{23}. \quad (12)$$

Analogically:

$$d^2E_2 = d\omega_{21}E_1 - \omega_{21} \wedge dE_1 + \\ + d\omega_{23}E_3 - \omega_{23} \wedge dE_3 = 0,$$

$$d\omega_{21}E_1 - \omega_{21} \wedge (\omega_{12}E_2 + \omega_{13}E_3) + \\ + d\omega_{23}E_3 - \omega_{23} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0,$$

$$(d\omega_{23} - \omega_{21} \wedge \omega_{13})E_3 + \\ +(d\omega_{21} - \omega_{23} \wedge \omega_{31})E_1 = 0. \quad (13)$$

From (13) we have

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}. \quad (14)$$

Equations (8), (9), (10), (12) and (14) are called Maurer-Cartan structural equations. From equation (1)0 and Cartan's lemma we have

$$\omega_{13} = \alpha_{11}\theta_1 + \alpha_{12}\theta_2, \\ \omega_{23} = \alpha_{12}\theta_1 + \alpha_{22}\theta_2. \quad (15)$$

From (15) and (12) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = \\ = -\omega_{13} \wedge \omega_{23} = \\ = -(\alpha_{11}\theta_1 + \alpha_{12}\theta_2) \wedge (\alpha_{12}\theta_1 + \alpha_{22}\theta_2). \quad (16)$$

Equation (16) gives

$$d\omega_{12} = -(\alpha_{11}\alpha_{22} - \alpha_{12}^2)\,\theta_1 \wedge \theta_2 = -K\theta_1 \wedge \theta_2, \quad (17)$$

where $K = \alpha_{11}\alpha_{22} - \alpha_{12}^2$ is the Gaussian curvature.

Differentiating the equation $E_3 \cdot E_3 = 1$ we have

$$dE_3 \cdot E_3 = 0,$$

which means that dE_3 is a tangent vector, i.e. $dE_3 \in T_p(M)$. The mapping

$$W(\alpha x_u + \beta x_v) = -\alpha\frac{\partial E_3}{\partial u} - \beta\frac{\partial E_3}{\partial v}$$

is a linear mapping

$$W : T_p(M) \to T_p(M).$$

Example 1: Let $x(u, v) = (u, v, u \cdot v)$ be a parametrized utility surface in $\mathbb{R}^3$. We are going to construct Gaussian and Mean curvature.
Moving frame is

$$x_u = (1, 0, v), \\ x_v = (0, 1, u), \\ n \;\; = (-v, -u, 1).$$

Orthonormal frame is

$$E_1 = \frac{1}{\sqrt{1+v^2}}(1, 0, v), \\ E_2 = \frac{1}{\sqrt{1+v^2} \cdot \sqrt{1+u^2+v^2}}(-uv, 1+v^2, u), \\ E_3 = \frac{1}{\sqrt{1+u^2+v^2}}(-v, -u, 1).$$

From (1) follows

$$\theta_1 = \sqrt{1+v^2}\,du + \frac{uv}{\sqrt{1+v^2}}\,dv, \quad (18)$$

$$\theta_2 = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+v^2}}\,dv. \quad (19)$$

Further we have

$$dE_1 = \left(\frac{-v}{(1+v^2)^{\frac{3}{2}}}, 0, \frac{1}{(1+v^2)^{\frac{3}{2}}}\right)\,dv,$$

$$\omega_{12} = dE_1 \cdot E_2 = \frac{u}{(1+v^2)\sqrt{1+u^2+v^2}}\,dv.$$

Analogically we have

$$\omega_{13} = dE_1 \cdot E_3 = \frac{1}{\sqrt{1+v^2}\sqrt{1+u^2+v^2}}\,dv.$$

Further we have

$$\partial_u E_2 = \frac{\sqrt{1+v^2}}{(1+u^2+v^2)^{\frac{3}{2}}} \cdot (-v, -u, 1),$$

$$\partial_v E_2 = \frac{1}{(1+v^2)^{\frac{3}{2}} \cdot (1+u^2+v^2)^{\frac{3}{2}}}(E_{2v}^1, E_{2v}^2, E_{2v}^3),$$

where

$$E_{2v}^1 = -u(1+v^2)(1+u^2) + uv^2(1+u^2+v^2), \\ E_{2v}^2 = u^2v(1+v^2), \\ E_{2v}^3 = -uv\left[(1+u^2+v^2) + (1+v^2)\right].$$

$$\partial_u E_2 \cdot E_3 = \frac{\sqrt{1+v^2}}{1+u^2+v^2}, \\ \partial_v E_2 \cdot E_3 = \frac{-uv}{\sqrt{1+v^2}(1+u^2+v^2)},$$

and

$$\omega_{23} = dE_2 \cdot E_3 = \\ = \frac{\sqrt{1+v^2}}{1+u^2+v^2}du - \frac{uv}{\sqrt{1+v^2}(1+u^2+v^2)}dv.$$

Summarizing previous results we obtain:

$$\omega_{12} = -\omega_{21} = \frac{u}{(1+v^2)\sqrt{1+u^2+v^2}}\,dv,$$

$$\omega_{13} = -\omega_{31} = \frac{1}{\sqrt{1+v^2}\sqrt{1+u^2+v^2}}\,dv,$$

$$\omega_{23} = -\omega_{32} = \frac{\sqrt{1+v^2}}{1+u^2+v^2}\,du - \\ - \frac{uv}{\sqrt{1+v^2}(1+u^2+v^2)}\,dv,$$

$$\theta_1 = \sqrt{1+v^2}\,du + \frac{uv}{\sqrt{1+v^2}}\,dv,$$

$$\theta_2 = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+v^2}}\,dv.$$

From equations (18) and (19) follows

$$d\theta_1 = 0, \quad d\theta_2 = \frac{u}{\sqrt{1+v^2}\sqrt{1+u^2v^2}}\, du \wedge dv$$

and

$$\theta_1 \wedge \theta_2 = \sqrt{1+u^2v^2}\, du \wedge dv. \tag{20}$$

From (12) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = \frac{1}{(1+u^2+v^2)^{\frac{3}{2}}}\, du \wedge dv.$$

Thanks to (20) we have

$$du \wedge dv = \frac{1}{\sqrt{1+u^2+v^2}}\, \theta_1 \wedge \theta_2$$

$$d\omega_{12} = \frac{1}{(1+u^2+v^2)^2}\, \theta_1 \wedge \theta_2.$$

From (17) immediately follows that

$$K = -\frac{1}{(1+u^2+v^2)^2}, \tag{21}$$

which means that every point of studied surface is hyperbolic.

The Weingarten map gives

$$W(x_u) = -\partial_u E_3, \quad \text{and} \quad W(x_v) = -\partial_v E_3,$$

where

$$\partial_u E_3 = \frac{1}{(1+u^2+v^2)^{\frac{3}{2}}}(uv, -v^2-1, -u),$$
$$\partial_v E_3 = \frac{1}{(1+u^2+v^2)^{\frac{3}{2}}}(-1-u^2, uv, -v).$$

From the fact $W : T_p(M) \to T_p(M)$ follows

$$\begin{aligned} \partial_u E_3 &= \beta_{11} x_u + \beta_{12} x_v, \\ \partial_v E_3 &= \beta_{21} x_u + \beta_{22} x_v. \end{aligned} \tag{22}$$

After a short calculation we obtain

$$\beta_{11} = \frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}},$$
$$\beta_{12} = -\frac{1+v^2}{(1+u^2+v^2)^{\frac{3}{2}}},$$
$$\beta_{21} = -\frac{1+u^2}{(1+u^2+v^2)^{\frac{3}{2}}},$$
$$\beta_{22} = \frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}}.$$

From equations (22) follows that the mapping W can be described by the matrix

$$W = \frac{1}{(1+u^2+v^2)^{\frac{3}{2}}} \begin{pmatrix} -uv & 1+v^2 \\ 1+u^2 & -uv \end{pmatrix}.$$

Determinant

$$\det W = K = \frac{1}{(1+u^2+v^2)^3} \det \begin{pmatrix} -uv & 1+v^2 \\ 1+u^2 & -uv \end{pmatrix} =$$
$$= -\frac{1}{(1+u^2+v^2)^2},$$

as was given in (20) and the formula for mean curvature is

$$H = \frac{1}{2} trW = -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}}.$$

Example 2. Let $x(u,v) = (u, v, u^2v)$ be other parameterized utility surface in $\mathbb{R}^3$. Moving frame is

$$x_u = (1, 0, 2uv),$$
$$x_v = (0, 1, u^2),$$
$$n = (-2uv, -u^2, 1).$$

Orthonormal frame is

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{1+4u^2v^2}}, (1, 0, 2uv), \\ E_2 &= \frac{(-2u^3v, 1+4u^2v^2, u^2)}{\sqrt{1+4u^2v^2}\sqrt{1+4u^2v^2+u^4}} \\ E_3 &= \frac{1}{\sqrt{1+4u^2v^2+u^4}}(-2uv, -u^2, 1). \end{aligned} \tag{23}$$

Further we have

$$dE_1 = \partial_u E_1 du + \partial_v E_1 dv = \frac{1}{(1+4u^2v^2)^{\frac{3}{2}}} \cdot$$
$$\cdot \left[\left(-4uv^2, 0, 2v\right) du + \left(-4u^2v, 0, 2u\right) dv \right].$$

After a short calculation we obtain

$$\omega_{12} = dE_1 \cdot E_2 = \frac{(2u^2vdu + 2u^3dv)}{(1+4u^2v^2)\sqrt{1+4u^2v^2+u^4}}.$$

Analogically

$$\omega_{13} = dE_1 \cdot E_3 = \frac{(2vdu + 2udv)}{\sqrt{1+4u^2v^2}\sqrt{1+4u^2v^2+u^4}}.$$

From (23) follows

$$dE_3 = (\partial_u E_3)du + (\partial_v E_3)dv.$$

After a short calculation we obtain

$$\omega_{32} = dE_3 \cdot E_2 = \frac{\left[\left(-4u^3v^2 - 2u\right) du + 4u^4vdv\right]}{\sqrt{1+4u^2v^2}(1+4u^2v^2+u^4)}.$$

Summarizing the previous results we obtain

$$\omega_{12} = -\,\omega_{21} = \frac{2u^2vdu + 2u^3dv}{(1+4u^2v^2)\sqrt{1+4u^2v^2+u^4}},$$

$$\omega_{13} = -\,\omega_{31} = \frac{2vdu + 2udv}{\sqrt{1+4u^2v^2}\sqrt{1+4u^2v^2+u^4}},$$

$$\omega_{23} = -\,\omega_{32} = \frac{\left(4u^3v^2 + 2u\right) du - 4u^4vdv}{\sqrt{1+4u^2v^2}(1+4u^2v^2+u^4)},$$

The forms θ_1 and θ_2 are

$$\begin{aligned} \theta_1 &= \sqrt{1+4u^2v^2}du + \frac{2u^3v}{\sqrt{1+4u^2v^2}}dv, \\ \theta_2 &= \frac{\sqrt{1+4u^2v^2+u^4}}{\sqrt{1+4u^2v}}dv. \end{aligned} \tag{24}$$

From equations (16) and (24) we obtain

$$d\omega_{12} = \omega_{13} \ \wedge \ \ \omega_{32} = \frac{4u^2}{(1+4u^2v^2+u^4)^{\frac{3}{2}}} du \wedge dv =$$
$$= \frac{4u^2}{(1+4u^2v^2+u^4)^2} \theta_1 \wedge \theta_2,$$

from which follows that the Gaussian curvature has the form

$$K = -\frac{4u^2}{(1+4u^2v^2+u^4)^2}.$$

Conclusion

Two economical examples served as an illustration of Maurer-Cartan equations and we reached the following results:

1. The Gaussian and mean curvatures of the first surface are
$$K = -\frac{1}{(1+u^2+v^2)^2},$$
$$H = \frac{1}{2}\mathrm{tr}W = -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}}.$$

2. The Gaussian curvature of the second surface is
$$K = -\frac{4u^2}{(1+4u^2v^2+u^4)^2}.$$

References

Bureš, J., Kaňka, M., 1994, "Some conditions for a surface in E^4 to be a part of the sphere S^2", Mathematica Bohemica No. 4, pp. 367–371

Kaňka, M., 1995, "An example of basic structure equations for Riemannian Manifolds", Mundus Symbolicus 1995, 57–62

Cartan, É., 1952, "Oeuvres complètes – Partie I.", Gauthier-Villars, vol. 2

Cartan, É., 1953, "Oeuvres complètes – Partie II.", Gauthier-Villars, vol. 1

Cartan, É., 1953, "Oeuvres complètes – Partie II.", Gauthier-Villars, vol. 2

Cartan, É., 1955, "Oeuvres complètes – Partie III.", Gauthier-Villars, vol. 1

Cartan, É., 1955, "Oeuvres complètes – Partie III.", Gauthier-Villars, vol. 2

Sternberg, S., 1964, "Lectures on Differential Geometry", Prentice-Hall, INC.

Kobayashi, S., Nomizu, K., 1963, "Foundations of differential geometry", New York

Nomizu, K., 1956, "Lie groups and differential geometry", The mathematical society of Japan

Spivak, M., 1979, "Differential geometry", Berkeley