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USING OF CARTAN‘S MOVING FRAME METHOD IN DIFFERENTIAL GEOMETRY OF SURFACES

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ABSTRACT

The aim of this paper is to give basic geometrical characteristics of sphere and torus, parameterization of which is $x(u,v) = (x_1,(u,v), x_2(u,v), x_3(u,v))$, but our interest is concentrated mainly to Cobb-Douglas surfaces used in economics. We are going to study these functions as regular surfaces in R^3. Applying the method of Cartan moving frame we obtain geometrical description of Cobb-Douglas function used in economy, parameterization of which has the form $x(u,v)=(u,v,A\cdot u^{\alpha}\cdot v^{\beta})$, where $A = 1$, $u > 0$, $v > 0$ and $\alpha, \beta \in R$.

JEL CLASSIFICATION & KEYWORDS

■ C00 ■ ORTHONORMAL FRAME ■ TANGENT SPACE ■ DIFFERENTIAL FORMS ■ GAUSSIAN CURVATURE ■ MAURER-CARTAN EQUATIONS ■ CARTAN'S LEMMA

INTRODUCTION

Let $U \subset R^2$ be an open neighbourhood of a point $(u,v) \in U$ and $x : U \to R^3$ a regular map. A subset $M \subset R^3$ is called a regular two dimensional surface in R^3 if for each $p \in M$ there exist an open neighbourhood V of $p \in R^3$ and a map $x : U \to V \cap M$ of an open set $U \subset R^2$ onto $V \cap M$ such that x is a differentiable homeomorphism and the differential $dx_q : T_q(U) \to T_{x(q)}(M)$ is injective for all $q \in U$, $x(q) = p$. Then it is possible to choose in $x(U)$ an orthonormal moving frame $\{E_1, E_2, E_3\}$ in such a way that E_1, E_2 are tangent to $x(U)$ and E_3 is a non-vanishing normal to $x(U)$.

Basic equations

We first discuss the Cartan structural equations for a two-dimensional surface in R^3. Differentiating a map $x(u,v)$ we obtain

$$dx = x_u du + x_v dv,$$

where x_u, x_v are tangent vector fields. Let us denote moving frame $\{x_u, x_v, n\}$ where n is a normal vector field, and

$$N(u,v) = \frac{x_u \times x_v}{\|x_u \times x_v\|}$$

is a unit normal field. With respect to the orthonormal moving frame $\{E_1, E_2, E_3\}$ we define forms

$$\theta_i = E_i dx = E_i x_u du + E_i x_v dv,\ i = 1,2,3. \tag{1}$$

Since x_u and x_u are tangent to $x(U)$ we have $E_3 dx = N dx = 0$ which implies $\theta_3 = 0$.

Each vector $E_i : U \subset R^3 \to R^3$ is a differentiable function and the differential $dE_i : T_q(U) \to T_{x(q)}(M)$ is a linear map. We may write (using Einstein's notation) $dE_i = \omega_{ij} E_j$, where ω_{ij} are linear forms on R^3 and since E_i are differentiable ω_{ij} are nine differentiable forms. We have

$$\begin{pmatrix} dE_1 \\ dE_2 \\ dE_3 \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2}$$

Differentiating equation $E_i \cdot E_j = \delta_{ij}$ where δ_{ij} is the Kronecker's symbol, we obtain

$d E_i E_j + E_i\, d E_j = \omega_{ij} + \omega_{ji} = 0.$

Forms ω_{ij} are antisymmetric

$$\omega_{ii} = 0,\ \omega_{ij} = -\omega_{ji}. \tag{3}$$

From (2) and (3) follows

$$\begin{pmatrix} dE_1 \\ dE_2 \\ dE_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{4}$$

Forms dx and dE_i have vanishing exterior derivatives, which means

$d^2x = 0$ and $d^2E_i = 0$, where $i = 1,2,3$.

So we have

$$0 = d^2x = dE_1 \wedge \theta_1 + E_1\, d\theta_1 + dE_2 \wedge \theta_2 + E_2\, d\theta_2. \tag{5}$$

Substituting (4) into (5) we obtain

$$(\omega_{12}E_2 + \omega_{13}E_3) \wedge \theta_1 + E_1 d\theta_1 + (\omega_{21}E_1 + \omega_{23}E_3) \wedge \theta_2 + E_2 d\theta_2 = 0. \tag{6}$$

From (6) follows

$$(d\theta_1 + \omega_{21} \wedge \theta_2) E_1 + (d\theta_2 + \omega_{12} \wedge \theta_1) E_2 + (\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2) E_3 = 0. \tag{7}$$

The linear independence of vectors E_1, E_2, E_3 and equation (7) gives the following equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2, \tag{8}$$

$$d\theta_2 = \omega_{21} \wedge \theta_1, \tag{9}$$

$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2, \tag{10}$$

Exterior derivatives (4) gives:

$$0 = d^2E_1 = d\omega_{12}E_2 - \omega_{12} \wedge dE_2 + d\omega_{13}E_3 - \omega_{13} \wedge dE_3,$$
$$d\omega_{12}E_2 - \omega_{12} \wedge (\omega_{21}E_1 + \omega_{23}E_3) + \omega_{13}E_3 - \omega_{13} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0,$$

we have

$$(d\omega_{12} - \omega_{13} \wedge \omega_{32})E_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})E_3 = 0. \tag{11}$$

From (11) follows

$$d\omega_{12} = \omega_{13} \wedge \omega_{32},\ d\omega_{13} = \omega_{12} \wedge \omega_{23}. \tag{12}$$

Analogically we have

$$\begin{gathered} d^2E_2 = d\omega_{21}E_1 - \omega_{21} \wedge d\,E_1 + d\,\omega_{23}\,E_3 - \omega_{23} \wedge d\,E_3 = 0, \\ d\omega_{21}E_1 - \omega_{21} \wedge (\omega_{12}E_2 + \omega_{13}E_3) + \omega_{23}E_3 - \omega_{23} \wedge (\omega_{31}E_1 + \omega_{32}\,E_2) = 0, \\ (d\,\omega_{23} - \omega_{21} \wedge \omega_{13})\,E_3 + (d\,\omega_{21} - \omega_{23} \wedge \omega_{31})\,E_1 = 0. \end{gathered} \tag{13}$$

From (13) follows

$$d\omega_{23} = \omega_{21} \wedge \omega_{13},\ d\omega_{21} = \omega_{23} \wedge \omega_{31}. \tag{14}$$

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Equations (8), (9), (10), (12) and (14) are called Maurer-Cartan structural equations. From equation (10) and Cartan's lemma follows

$$\omega_{13} = \alpha_{11}\theta_1 + \alpha_{12}\theta_2, \quad \omega_{23} = \alpha_{12}\theta_1 + \alpha_{22}\theta_2. \tag{15}$$

From (15) and (12) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23}$$
$$= -(\alpha_{11}\theta_1 + \alpha_{12}\theta_2) \wedge (\alpha_{12}\theta_1 + \alpha_{22}\theta_2). \tag{16}$$

Equation (16) gives

$$d\omega_{12} = -\left(\alpha_{11}\alpha_{22} - \alpha_{12}^2\right)\theta_1 \wedge \theta_2 = -K\,\theta_1 \wedge \theta_2,$$

where $K = \alpha_{11}\alpha_{22} - \alpha^2_{12}$ is the Gaussian curvature.

Examples

Example 1. Sphere $S^2 \subset R^3$

Local parameterization of the sphere $S^2 \subset R^3$ is given by the map $x(u,v) = (r\cos v\cos u, r\cos v\sin u, r\sin v)$, where

$$(u,v) \in (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The moving frame is

$x_u = (-r\cos v\sin u, r\cos v\cos u, 0) = r\cos v(-\sin u, \cos u, 0)$,

$x_v = r(-\sin v\cos u, -\sin v\sin u, \cos v)$,

$n = r^2 \cdot \cos v(\cos u\cos v, \sin u\cos v, \sin v)$.

Orthonormal moving frame is

$E_1 = (-\sin u, \cos u, 0)$,

$E_2 = (-\sin v\cos u, -\sin v\sin u, \cos v)$,

$E_3 = (\cos u\cos v, \sin u\cos v, \sin v)$.

The differential dE_1 equals

$\partial_u E_1 = (-\cos u, -\sin u, 0)$,

$\partial_v E_1 = (0, 0, 0)$,

$dE_1 = (-\cos u, -\sin u, 0)du$.

Forms ω_{12} and ω_{13} are

$$\omega_{12} = dE_1 \cdot E_2 = (\sin v\cos^2 u + \sin v\sin^2 u)\,du = \sin v\,du,$$
$$\omega_{13} = dE_1 \cdot E_3 = (-\cos v\cos^2 u - \cos v\sin^2 u)\,du = -\cos v\,du.$$

Analogically we have

$$\partial_u E_2 = (\sin v\sin u, -\sin v\cos u, 0),$$
$$\partial_v E_2 = (-\cos v\cos u, -\cos v\sin u, -\sin v).$$
$$dE_2 = (\sin v\sin u, -\sin v\cos u, 0)\,du + (-\cos v\cos u, -\cos v\sin u, -\sin v)\,dv.$$

Forms ω_{21} and ω_{23} are

$$\omega_{21} = dE_2 \cdot E_1 = \left(-\sin v\sin^2 u - \sin v\cos^2 u\right)du + \left(\sin u\cos v\cos u - \cos u\cos v\sin u\right)dv$$
$$= -\sin v\,du,$$
$$\omega_{23} = dE_2 \cdot E_3 = \left(\sin v\sin u\cos u\cos v - \sin v\cos u\sin u\cos v\right)du$$
$$+\left(-\cos v\cos u \cdot \cos u\cos v - \cos v\sin u\sin u\cos v - \sin^2 v\right)dv$$
$$= \left(-\cos^2 v\cos^2 u - \cos^2 v\sin^2 u - \sin^2 v\right)dv$$
$$= \left(-\cos^2 v(\cos^2 u + \sin^2 u) - \sin^2 v\right)dv = -dv.$$

The differential dE_3 equals

$$\partial_u E_3 = (-\sin u\cos v, \cos u\cos v, 0),$$
$$\partial_v E_3 = (-\cos u\sin v, -\sin u\sin v, \cos v),$$
$$dE_3 = (-\sin u\cos v, \cos u\cos v, 0)\,du + (-\cos u\sin v, -\sin u\sin v, \cos v)\,dv.$$

Forms ω_{31} and ω_{32} are

$$\omega_{31} = dE_3 \cdot E_1 = \left(\sin^2 u\cos v + \cos^2 u\cos v\right)du +$$
$$+\left(\sin u\cos u\sin v - \sin u\sin v\cos u\right)dv = \cos v\,du,$$

$$\omega_{32} = dE_3 \cdot E_2 = \left(\sin u\cos v\sin v\cos u - \cos u\cos v\sin v\sin u\right)du +$$
$$+\left(\cos^2 u\sin^2 v + \sin^2 u\sin^2 v + \cos^2 v\right)dv = dv.$$

The forms θ_1 and θ_2 are

$$\theta_1 = E_1(x_u du + x_v dv) =$$
$$= (-\sin u, \cos u, 0)(-r\cos v\sin u, r\cos v\cos u, 0)\,du$$
$$+ (-\sin u, \cos u, 0)(-r\sin v\cos u, -r\sin v\sin u, r\cos v)\,dv$$
$$= \left(r\sin^2 u\cos v + r\cos^2 u\cos v\right)du$$
$$+ \left(r\sin u\cos u\sin v - r\sin v\sin u\cos u\right)dv$$
$$= r\cos v\,du,$$

$$\theta_2 = E_2(x_u du + x_v dv) =$$
$$= (-\sin v\cos u, -\sin v\sin u, \cos v)(-r\sin u\cos v, r\cos u\cos v, 0)\,du$$
$$+ (-\sin v\cos u, -\sin v\sin u, \cos v)(-r\cos u\sin v, -r\sin u\sin v, r\cos v)\,dv$$
$$= \left(r\sin v\cos u\sin u\cos v - r\sin v\sin u\cos u\cos v\right)du$$
$$+ \left(r\sin^2 v\cos^2 u + r\sin^2 v\sin^2 u + r\cos^2 v\right)dv$$
$$= r\,dv.$$

From exterior product $\theta_1 \wedge \theta_2 = r^2\cos v\,du \wedge dv$ follows

$$du \wedge dv = \frac{1}{r^2\cos v}\,\theta_1 \wedge \theta_2.$$

Further we have $\omega_{31} \wedge \omega_{32} = \cos v\,du \wedge dv$, and

$$\omega_{31} \wedge \omega_{32} = \cos v\frac{1}{r^2\cos v}\theta_1 \wedge \theta_2 = \frac{1}{r^2}\theta_1 \wedge \theta_2.$$

The equation $\omega_{31} \wedge \omega_{32} = K \cdot \theta_1 \wedge \theta_2$, where K is Gaussian curvature gives

$$K = \frac{1}{r^2}.$$

Example 2. Torus $T^2 \subset R^3$

Local parameterization of the torus in R^3 is given by the map $x(u,v) = ((a + b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v)$, where

$$a > b > 0,\ u \in (0, 2\pi),\ v \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

The moving frame has the form

$$x_u = (a + b\cos v)(-\sin u, \cos u, 0),$$
$$x_v = b(-\sin v\cos u, -\sin v\sin u, \cos v),$$
$$N = (\cos u\cos v, \sin u\cos v, \sin v).$$

The orthonormal frame can be written in the form

$$E_1 = (-\sin u, \cos u, 0),$$
$$E_2 = (-\sin v\cos u, -\sin v\sin u, \cos v),$$
$$E_3 = (\cos u\cos v, \sin u\cos v, \sin v).$$

$$\partial_u E_1 = (-\cos u, -\sin u, 0),$$
$$\partial_v E_1 = (0,0,0).$$

So $dE_1 = (-\cos u, -\sin u, 0)\,du$.

Forms ω_{12} and ω_{13} are

$$\omega_{12} = dE_1 \cdot E_2 =$$
$$= (-\cos u, -\sin u, 0)\,du \cdot (-\sin v\cos u, -\sin v\sin u, \cos v)$$
$$= \left(\sin v\cos^2 u + \sin v\sin^2 u + 0\right)du = \sin v\,du,$$
$$\omega_{13} = dE_1 \cdot E_3 =$$
$$= (-\cos u, -\sin u, 0) \cdot (\cos u\cos v, \sin u\cos v, \sin v)\,du$$
$$= -\cos^2 u\cos v - \sin^2 u\cos v = -\cos v\,du.$$

Further we have

$$\theta_1 = E_1(x_u du + x_v dv) =$$
$$= (-\sin u, \cos u, 0) \cdot \left[(a + b\cos v)(-\sin u, \cos u, 0)\,du + b(-\sin v\cos u, -\sin v\sin u, \cos v)\,dv\right]$$
$$= (a + b\cos v)du + b\left[\sin u\cos u\sin v - \sin u\cos u\sin v + 0\right]dv$$
$$= (a + b\cos v)\,du,$$

$$
\begin{aligned}
\theta_2 &= E_2(x_u du + x_v dv) = \\
&= (-\sin v\cos u, -\sin v\sin u, \cos v)\cdot \\
&\cdot [(a+b\cos v)\cdot(-\sin u, \cos u, 0)\,du + b(-\sin v\cos u, -\sin v\sin u, \cos v)\,dv] \\
&= (a+b\cos v)[\sin v\cos u\sin u - \sin v\sin u\cos u + 0]du \\
&+ b\left[\sin^2 v\cos^2 u + \sin^2 v\sin^2 u + \cos^2 v\right]dv \\
&= b\left[\sin^2 v(\cos^2 u + \sin^2 u) + \cos^2 v\right]dv = b\,dv.
\end{aligned}
$$

We have

$$\omega_{12} = \sin v\,du,$$

$$\theta_1 = (a+b\cos v)du, \quad \theta_2 = b\,dv,$$

$$\theta_1 \wedge \theta_2 = b(a+b\cos v)du \wedge dv,$$

$$du \wedge dv = \frac{1}{b(a+b\cos v)}\theta_1 \wedge \theta_2,$$

$$d\omega_{12} = -\cos v\,du \wedge dv,$$

$$d\omega_{12} = -\frac{\cos v}{b(a+b\cos v)}\,\theta_1 \wedge \theta_2.$$

The equation $d\omega_{12} = -K\cdot\theta_1 \wedge \theta_2$, gives the formula for Gaussian curvature K

$$K = \frac{\cos v}{b(a+b\cos v)}.$$

Example 3. Gaussian curvature of Cobb-Douglas surfaces

Let $x(u,v)=(u,v,u^\alpha \cdot v^\beta)$, where $u > 0$, $v > 0$ and $\alpha, \beta \in R$ are studied Cobb Douglas surfaces:

$$x(u,v) = (u, v, u^\alpha \cdot v^\beta),$$

$$(u^\alpha \cdot v^\beta)_u = \alpha \cdot u^{\alpha-1} \cdot v^\beta,$$

$$(u^\alpha \cdot v^\beta)_v = \beta \cdot u^\alpha \cdot v^{\beta-1},$$

The moving frame is

$$x_u = (1,\ 0,\ \alpha \cdot u^{\alpha-1} \cdot v^\beta),$$

$$x_v = (0,\ 1,\ \beta \cdot u^\alpha \cdot v^{\beta-1}),$$

$$n = (-\alpha \cdot u^{\alpha-1} \cdot v^\beta,\ -\beta \cdot u^\alpha \cdot v^{\beta-1},\ 1).$$

Let us denote

$$A = 1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta},$$

$$B = 1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2}.$$

Orthonormal moving frame has the form

$$E_1 = \left(\frac{1}{\sqrt{A}},\ 0,\ \frac{\alpha \cdot u^{\alpha-1} \cdot v^\beta}{\sqrt{A}}\right),$$

$$E_2 = \left(\frac{-\alpha\beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}},\ \frac{A}{\sqrt{A}\sqrt{B}},\ \frac{\beta \cdot u^\alpha \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}}\right),$$

$$E_3 = \left(\frac{-\alpha \cdot u^{\alpha-1} \cdot v^\beta}{\sqrt{B}},\ \frac{-\beta \cdot u^\alpha \cdot v^{\beta-1}}{\sqrt{B}},\ \frac{1}{\sqrt{B}}\right).$$

We have

$$\partial_u E_1 = \left(\frac{-\alpha^2 \cdot (\alpha-1) \cdot u^{2\alpha-3} \cdot v^{2\beta}}{A^{3/2}},\ 0,\ \frac{\alpha \cdot (\alpha-1) \cdot u^{\alpha-2} \cdot v^\beta}{A^{3/2}}\right),$$

$$\partial_v E_1 = \left(\frac{-\alpha^2 \cdot \beta \cdot u^{2\alpha-2} \cdot v^{2\beta-1}}{A^{3/2}},\ 0,\ \frac{\alpha \cdot \beta \cdot u^{\alpha-1} \cdot v^{\beta-1}}{A^{3/2}}\right),$$

$$dE_1 = \partial_u E_1\,du + \partial_v E_1\,dv,$$

$$\omega_{12} = dE_1 \cdot E_2 = \frac{\left[\alpha \cdot (\alpha-1) \cdot \beta \cdot u^{2\alpha-2} \cdot v^{2\beta-1}\right]du + \left[\alpha \cdot \beta^2 \cdot u^{2\alpha-1} \cdot v^{2\beta-2}\right]dv}{A\sqrt{B}},$$

$$\omega_{31} = dE_3 \cdot E_1 = \frac{-\left[\alpha \cdot (\alpha-1) \cdot u^{\alpha-2} \cdot v^\beta\right]du - \left[\alpha \cdot \beta \cdot u^{\alpha-1} \cdot v^{\beta-1}\right]dv}{\sqrt{A} \cdot \sqrt{B}}.$$

Analogically we obtain

$$\omega_{32} = \frac{\alpha^2 \cdot (\alpha-1) \cdot \beta \cdot u^{3\alpha-3} \cdot v^{3\beta-1} - \alpha \cdot \beta \cdot u^{\alpha-1} \cdot v^{\beta-1} \cdot A}{\sqrt{A}\,B}\,du + $$

$$+ \frac{\alpha^2 \cdot \beta^2 \cdot u^{3\alpha-2} \cdot v^{3\beta-2} - \beta \cdot (\beta-1) \cdot u^\alpha \cdot v^{\beta-2} \cdot A}{\sqrt{A}\,B}\,dv.$$

The exterior product of ω_{31} and ω_{32} is

$$
\begin{aligned}
\omega_{31} \wedge \omega_{32} &= \frac{-\alpha^3 \cdot (\alpha-1) \cdot \beta^2 \cdot u^{4\alpha-4} \cdot v^{4\beta-2} + \alpha \cdot \beta \cdot (\alpha-1) \cdot (\beta-1) \cdot u^{2\alpha-2} \cdot v^{2\beta-2} \cdot A}{A \cdot B^{3/2}}\,du \wedge dv \\
&+ \frac{\alpha^3 \cdot (\alpha-1) \cdot \beta^2 \cdot u^{4\alpha-4} \cdot v^{4\beta-2} - \alpha^2 \cdot \beta^2 \cdot u^{2\alpha-2} \cdot v^{2\beta-2} \cdot A}{A \cdot B^{3/2}}\,du \wedge dv \\
&= \frac{\alpha \cdot \beta \cdot (\alpha-1) \cdot (\beta-1) \cdot u^{2\alpha-2} \cdot v^{2\beta-2} - \alpha^2 \cdot \beta^2 \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}}\,du \wedge dv \\
&= \frac{\alpha \cdot \beta \cdot (\alpha \cdot \beta - \alpha - \beta + 1 - \alpha \cdot \beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}}\,du \wedge dv \\
&= \frac{\alpha \cdot \beta \cdot (-\alpha - \beta + 1) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}}\,du \wedge dv.
\end{aligned}
$$

Further we have $\theta_i = E_i x_u du + E_i x_v dv$, $i = 1,2$.

Specially

$$
\begin{aligned}
\theta_1 &= E_1 x_u du + E_1 x_v dv \\
&= \left(\frac{1}{\sqrt{A}}, 0, \frac{\alpha \cdot u^{\alpha-1} \cdot v^\beta}{\sqrt{A}}\right)(1, 0, \alpha \cdot u^{\alpha-1} \cdot v^\beta)\,du + \\
&+ \left(\frac{1}{\sqrt{A}}, 0, \frac{\alpha \cdot u^{\alpha-1} \cdot v^\beta}{\sqrt{A}}\right)(0, 1, \beta \cdot u^\alpha \cdot v^{\beta-1})\,dv \\
&= \sqrt{A}\,du + \frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}}\,dv.
\end{aligned}
$$

Analogically we obtain

$$
\begin{aligned}
\theta_2 &= E_2 x_u du + E_2 x_v dv = \\
&= \left(-\frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}}, \frac{1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta}}{\sqrt{A}\sqrt{B}}, \frac{\beta \cdot u^\alpha \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}}\right)(1, 0, \alpha \cdot u^{\alpha-1} \cdot v^\beta)\,du + \\
&+ \left(-\frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}}, \frac{1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta}}{\sqrt{A}\sqrt{B}}, \frac{\beta \cdot u^\alpha \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}}\right)(0, 1, \beta \cdot u^\alpha \cdot v^{\beta-1})\,dv \\
&= \frac{1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2}}{\sqrt{A}\sqrt{B}}\,dv.
\end{aligned}
$$

Finally we have

$$\theta_1 \wedge \theta_2 = \sqrt{A}\,\frac{\sqrt{B}}{\sqrt{A}}\,du \wedge dv,$$

which means

$$du \wedge dv = \frac{1}{\sqrt{B}}\theta_1 \wedge \theta_2.$$

The final result is

$$\omega_{31} \wedge \omega_{32} = \frac{\alpha \cdot \beta \cdot (1 - \alpha - \beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^2}\theta_1 \wedge \theta_2.$$

The Gaussian curvature equals

$$K = \frac{\alpha \cdot \beta \cdot (1 - \alpha - \beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{(1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2})^2}.$$

Conclusion

Two economical examples served as an illustration of Maurer-Cartan equations and we reached the following results.

The Gaussian curvatures of the studied surfaces are:

Example 1: $K = \frac{1}{r^2}$.

Example 2: $K = \frac{\cos v}{b \cdot (a + b\cos v)}$.

Example 3: $K = \frac{\alpha \cdot \beta \cdot (1 - \alpha - \beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{(1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2})^2}$.

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